FUOYE Journal of Pure and Applied Sciences

Available online at www.fuoye.edu.ng

OPTIMAL INVESTMENT STRATEGY FOR AN INVESTOR WITH ORNSTEIN-UHLENBECK AND CONSTANT ELASTICITY OF VARIANCE (CEV) MODELS UNDER CORRELATING AND NON-CORRELATING BROWNIAN MOTION

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ARTICLE INFO

Received: 18 February 2020
Accepted: 22 May 2020

Keywords: *Hamilton-Jacobi-Bellman equation, maximum principle, utility, constant elasticity of variance, Ornstein-Uhlenbeck process.*

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Abstract

This paper is intended to determine the optimal investment strategy of an investor operating in the financial market where the interest rate of the risk-free asset is governed by a stochastic model and the risky asset is assumed to follow constant elasticity of variance model, in particular we examine the event where the Brownian motions correlates and does not correlate. The main objective of the paper is to obtain an optimal investment policy for an investor who faces power utility preference and investigate the deviations which occur when the Brownian motions correlate and do not correlate. The optimal investment strategy for the value function is defined as $G(V,t;T) = Max_{\pi(t)}E[U(V)]$ *subject to:* $dV(t) = \{(\mu - r(t))\pi(t) + r(t)V(t)\}dt + bS^{\gamma}(t)\pi(t)dZ_1(t)$ *was constructed where* $r(t)$ *is the risk free rate,* $\pi(t)$ *is the amount invested in risky asset,* $dZ_1(t)$ *is the increase in the Wiener process,* (μ, b) *is the expected return and volatility of the stock market respectively and* V *is the utility preference and consequently applying Hamilton-Jacobi-Bellman equation and maximum principle technique to solve the power utility optimal investment strategy problem for an investor under constant elasticity of variance, we obtained a closed form solution* $\pi^* = \left[\frac{(\mu-r)}{cb^2S^{2\gamma}} + \frac{(1-c)}{c}\right]V$ *to the optimal investment strategies. This result suggests that the amount invested in risky assets is proportional to total utility amount of money available for investment.*

1.0 Introduction

Issues in financial asset allocation problems in discrete and continuous time are among many problems evaluated in actuarial finance literature and can be traced to [1, 2]. The author solved the investor's expected utility maximization problem when investing in stock and considering consumption where the underlying asset assumes Black-Scholes model under particular utility preference function. Markowitz originated the modern portfolio theory which is anchored on an assumption that the investing public directs their strength towards minimizing risk but focus to achieve the largest possible optimum return. His theory proves that it is possible for differing portfolios to have different levels of risk and return for an investor to decide which level of risk they are willing to incure to enable them diversify their portfolios based on the results of the decision taken. An investor may react rationally within these parameter framework and taking decisions hoping to maximize return under an achievable level of acceptable uncertainty. An investor can decide the quantum of volatility he wants to assume in his alternative portfolio by selecting the level that lies parallel to the efficient frontier so as to produce maximum return opportunities for the volume of risk which the investor has assumed despite the fact that it seems tasking to optimize a portfolio in real terms. The examination of optimal portfolios may be complemented by approximating different expected value of returns a number of times for each amount of risk. The optimal portfolio problem of utility maximization remains an indelible model in theory of interest mathematics and this has motivated many scholars to employ stochastic control technique to analyze investment problem. The martingale technique was formulated in [3] for investment problems of utility maximization using the martingale approach for an incomplete market, where the price process of risky assets is assumed to follow the geometric Brownian motion implying that the volatility of risky assets is constant but deterministic which empirical evidence has not supported and hence, it has become apparent that a stochastic volatility model is more realistic. The constant elasticity of variance model is a stochastic volatility formula but an extension of the geometric Brownian motion model with a view to capturing the implied volatility.

The concept of the constant elasticity of variance model was extended by [4] into life insurance annuity policies and the optimal investment strategies within utility function framework using dynamic programming principle. An investor wishes to maximize the expected utility of terminal wealth where he is permitted to invest in a risk-less asset and a risky asset. By applying stochastic optimal control, the Hamilton-Jacobi-Bellman (HJB) equation in conjunction with the help of the maximum principle can be transformed into a complex non-linear partial differential equation. Because of the difficulty level of the solution characterization, power transform and variable change technique to simplify the partial differential equation is used to obtain an explicit solution of the investor's problem.

The transformation of the non-linear second-order partial differential equation into a linear type through elimination of dependency on the wealth variable w and the price variable π of the risky asset was demonstrated in [5]. Within the continuous time framework of financial application, the state variable in the stochastic differential equation is wealth while the controls are the shares placed at each time in differing assets. Given the asset allocation chosen at a time, the determinants of the change in wealth are the stochastic returns on assets and the interest rate on the risk-less asset.

2.0 Some Related Literature

We observe that $[1-2,6]$ pioneered the problem of utility maximization which model was employed to assess the credit risk of a firm's indebtedness. Market experts and investors use the Merton model to assess how efficient will a firm fulfill the requirements of financial obligations to service its debt and measure the probability that it will go into credit default. Consequently, the model was developed by [7]. However, in [1] classical portfolio optimization problem, an investor can divide his wealth into a risky asset and a risk-less asset and then choose an optimal consumption rate to maximize total expected discounted utility of consumption. The constant elasticity of variance model which has the capacity of capturing the implied skewness was suggested in [8], the model which is empirically tractable in comparison with other stochastic volatility models. While solving an investment and consumption problem with stochastic interest rate where rate of interest followed the Ho-lee model and allowed to correlate with stock price, optimal strategies for power and logarithmic utility function was obtained in [9]

The motivation to solve the portfolio problem of a pension scheme management in a complete financial market with stochastic interest rate was demonstrated in [4]. We observed comparisons of two financial markets one risk-free and the other with risky asset under the application of geometric Brownian motion in [10]. The goal was to maximize the cumulative expected utility problem of consumption over certain planning time horizon and hence formulated stochastic impulse control problem where an investor with a bank account has a chance to transfer money between two assets and assumes that these transfers involve fixed transaction together with its cost proportional to the size of transaction.

We noticed in [2] the solution of the optimal time continuous allocation problem under risk uncertainty where the process of the risky assets is widely correlated with geometric Brownian motion under the assumption that the portfolio would be rebalanced momentarily with no cost. The goal was to maximize the net expected utility of consumption and the utility of terminal wealth. The author kept the fractions invested in the risky asset equal to a constant vector and assumed a consumption rate proportional to the total wealth by applying an optimal trading strategy with an infinite number of transactions and utility function in the constant relative risk aversion. The application of constant elasticity of variance model to solve utility maximization portfolio selection problem with multiple risky assets and a risk-free asset and simultaneously obtaining the Hamilton–Jacobi–Bellman equation associated with the portfolio optimization problem was demonstrated in [11]. Through power transformation technique and a variable change approach, an explicit solution for the constant absolute risk aversion utility function was obtained given the elasticity coefficient as -1 or 0. While attempting to evaluate the most adequate optimal strategy to all values of the elasticity coefficient, a model with two risky assets and one risk-free asset was suggested under a defined assumption to enable them analyze the characteristics of optimal strategies with a numerical simulation proven to contrast the results of the two models suggested. In $[12-13]$, optimal control of excess-of-loss Reinsurance and Investment problem for Insurers was examined and solved Under Constant Elasticity of variance (CEV) model Stock price assumed to follow constant elasticity of variance model while investigating an investor's portfolio problem where consumption, taxes, transaction costs and dividends are involved under constant elasticity of variance was modeled in [14]. The authors maximized the expected utility of consumption and terminal wealth under power utility preference. The application of the maximum principle assisted to find the Hamilton-Jacobi-Bellman equation for the value function where dependency of variable was eliminated to arrive at a close form solution of the optimal investment and consumption strategies. It was discovered that optimal investment on the risky asset is time horizon dependent. Furthermore, in $[15-16]$ while applying the maximum principle to Hamilton-Jacobi-Bellman equation for the value function provided a closed form solution to an investment and consumption decision problem where the risk-free asset has a rate of return that is assumed to follow the Ornstein-Uhlenbeck stochastic interest rate of return model. Arising from the theory of the consumption factor and the Ornstein-Uhlenbeck stochastic interest rate of return, the Hamilton-Jacobi-Bellman equation obtained becomes more challenging. The nonlinear second-order partial differential equation was transformed into an ordinary differential equation, the Bernoulli equation by using elimination of dependency of variables which provided the solution.

3.0 Geometric Brownian Motions

The Brownian motion transforms is a technically powerful tool in financial modeling such as modeling the evolution in time of stock prices and options, a market instrument which describes financial contracts that gives the holder the right but not the obligation to purchase or sell an underlying asset at a fixed, predetermined strike price on or before the maturity date. Options are usually characterized by the nature of the interval during which it is exercised. The option which is exercised at a period between the initial time of contract and the maturity date is described as American option while European option is only exercised at the maturity date. Two reasons account for why economic agents use options are speculation when the buyer knows that the value of the asset fluctuates up or down by the time the option is exercised and hedging which is used to mitigate the risk connected to the fluctuations in the price of the asset Geometric Brownian motion is a continuous-time stochastic process where the logarithm of the randomly varying quantity assumes a Brownian motion with drift. It is a stochastic process satisfying a stochastic differential equation applicable in mathematical finance to model stock prices. A stochastic process $S(t)$ is said to follow Geometric Brownian motion provided it satisfies the following stochastic differential equation $dS(t) = \mu S(t)dt + \sigma S(t)dZ(t)$,

where $Z(t)$ is a Brownian motion, μ appreciation rate and σ volatility are constants. Brownian motion is a very powerful vehicle for modeling continuous time stochastic process that is widely used in finance to model random event that occurs in an interval of time. It has become useful in modeling stock markets such as fluctuations in an asset's price. The path of Brownian motion is in continuum but it is nowhere smooth and consequently making it nowhere differentiable and hence when a particle assumes the trajectory of Brownian motion, it is impossible for it to jump from one point to another in instantaneous time.

i) $W(0) = 0$, the process always starts at 0
ii) With probability1, the function $t \to W(t)$ is continuous in $t > 0$ with

continuous sample path with no jump discontinuities

iii) $W(t)$ has stationary and independent increments.

iv) The increment $W(t+s)-W(s)\sim N(0,\sigma^2(t))$

$N(0,\sigma^2(t))$ denotes the normal distribution with the expected value μ, and variance σ^2.

We observe two consequences of the definition of Brownian motion

3.1 Theorem 1

(i) $W(t+s-r)=W(t+s)-W(r)$ are equal in distribution.

and (ii) $E\big(W(t+s)\big)-E\big(W(r)\big)=0$

Proof

The condition that it has independent increments means that if $0\le s_1\le t_1\le s_2\le t_2\le,\dots,\le s_{k-1}\le t_{k-1}\le s_k\le t_k$ then $W(t_1)-W(s_1),W(t_2)-W(s_2),\dots,W(t_{k-1})-W(s_{k-1}),W(t_k)-W(s_k)$ are independent random variables.

We observe that $W(t+s)=W(t+s)-W(0)\sim N\big(0,\sigma^2(t+s-0)\big)$ implying that

$W(t+s)\sim N\big(0,\sigma^2(t+s)\big)$. Furthermore

$W(t+s-r)\sim N\big(0,\sigma^2(t+s-r)\big)$

Now, $W(t+s-r)-W(0)\sim N\big(0,\sigma^2(t+s-r)\big)$, since $W(r)=0,\ r=0$

But by definition, $W(t+s)-W(r)\sim N\big(0,\sigma^2(t+s-r)\big)$,

hence $W(t+s-r)=W(t+s)-W(r)$ in distribution and,

$E\big(W(t+s)-W(r)\big)=E\big(W(t+s)\big)-E\big(W(r)\big)=0$.

As a consequence of the above theorem, we have the following

3.2 Theorem 2

$E\big(B(v+r)|B(u)\big)=B(v)$

$\sigma B(t)=W(t)$ for which the function $B(t)$ describes the standard Brownian motion.

Let $u\in[0,v], E\big(B(v+r)|B(u)\big)=E\big(B(v+r)|B(v)\big), r\ge 0$

$E\big(B(v+r)|B(u)\big)=E\big(B(v+r)|B(v)\big)=E\big(B(v)+[B(v+r)-B(v)]|B(v)\big)$

$E\big(B(v+r)|B(u)\big)=E\big(B(v)\big)+E[B(v+r)-B(v)]|B(v)$

$E\big(B(v+r)|B(u)\big)=B(v)+E[B(v+r)-B(v)]|B(v)$

$E\big(B(v+r)|B(u)\big)=B(v)+E[B(v+r)-B(v)]$

$E\big(B(v+r)|B(u)\big)=B(v)+E[B(v+r-v)]$

$E\big(B(v+r)|B(u)\big)=B(v)+E[B(r)]$, setting $(\text{setting to its minimum, that is}, r=0, \text{we have})$

$E\big(B(v+r)|B(u)\big)=B(v)\sim N\big(0,\sigma^2(v)\big)$

4.0 Ito's Integral Function:

Let $f(y,s)$ be a function where the partial derivatives f_s, f_y, f_{ss} are defined and continuous.

Define the diffusion process $dY_s=\mu(Y_s,s)ds+\sigma(Y_s,s)dW_s, 0\le s\le S$

$$f(Y_S,S)=f(Y_0,0)+\int_0^S\frac{\partial f(Y_S,S)}{\partial s}ds+\int_0^S\frac{\partial f(Y_S,S)}{\partial y}ds+\frac{1}{2}\int_0^S\frac{\partial^2 f(Y_S,S)}{\partial y^2}dY_SdY_S.$$

$$f(Y_S, S) = f(Y_0, 0) + \int_0^S \frac{\partial f(Y_S,S)}{\partial s} + \mu(Y_s, s)\frac{\partial f(Y_S,S)}{\partial s} + \frac{1}{2}\sigma^2(Y_s, s)\frac{\partial^2 f(Y_S,S)}{\partial y^2} ds + \int_0^S \sigma(Y_s, s)\frac{\partial f(Y_S,S)}{\partial y} dW_S$$

Therefore, $df(Y_S, S) = \frac{\partial f(Y_S,S)}{\partial s} ds + \frac{1}{2}\sigma^2(Y_s, s)\frac{\partial^2 f(Y_S,S)}{\partial y^2} ds + \mu(Y_s, s)\frac{\partial f(Y_S,S)}{\partial s} dW_S$

5.0 Ornstein-Uhlenbeck Model

The Ornstein-Uhlenbeck process is one of several approaches used to model interest rates, currency, exchange rates and commodity prices stochastically. An Ornstein-Uhlembeck process $r(t)$ satisfies the following stochastic differential equation:

$dr(t) = \theta(\mu - r(t))dt + \sigma dZ(t)$ where $\theta > 0$ *and* $\sigma > 0$ are parameters and $Z(t)$ denotes the Wiener process. It is also known as the Vasicek model. The parameter μ represents the equilibrium or mean value supported by fundamentals, σ, the degree of volatility around it caused by shocks, and θ the rate by which these shocks dissipate and the variable reverts towards the mean.

6.0 Methodology: Constant Elasticity Of Variance (CEV) Model

The constant elasticity of variance model is a stochastic volatility model that captures stochastic variability and the leveraging consequences. It is a tool popularly applied by market experts in the financial market in particular when stocks and bonds are modeled.

It is a process that is anchored on the following stochastic differential equation:

$dS(t) = S(t)[\mu dt + \sigma S(t)^{\gamma} dZ(t)]$ where $S(t)$ is the spot price, t is time, and μ is a parameter characterizing the drift, σ and γ are other parameters and Z is a Brownian motion, $dS(t)$ defines a differential, the constant parameters σ and γ meet the condition $\sigma \geq 0, \gamma \geq 0$, while γ controls the correspondence between variability and price.

The leverage effect which usually is noticed in stock markets occurs when $\boldsymbol{\gamma} < 1$ during which time the volatility of a stock increases as its price falls. However, in commodity markets, there is inverse leverage effect where the volatility of the price of a commodity seems to increase when its price increases and hence $\boldsymbol{\gamma} > 1$.

7.0 Maximum Principle & The Hamilton-Jacobi-Bellman (HJB) Equation

Dynamic programming is a maximization and minimization method concerned with synthesizing complex problems by breaking it down into constituent sub-problems through a recursive form. Naturally some decisions problems may not break down this manner but decisions which cover several horizons in time interval usually break recursively. Typical optimization problems have some objectives such as maximizing profit, maximizing output, maximizing speed, minimizing travel time, minimizing cost. The mathematical function which specifies this objective is the value or objective function. Maximum principle is employed in optimal control theory to obtain a possible control for moving a dynamical system from a position to another one subject to given constraints in the presence of state or input controls. The maximum or minimum of the extreme value function is both dependent on the problem and on the convention of sign applicable in describing the Hamiltonian. If u is the set of values of permissible control, then the principle states that the optimal control u^* must satisfy:

$H(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t) \leq H(\mathbf{x}^*(t), u, \boldsymbol{\lambda}^*(t), t), \forall\, u \in U, t \in [t_\mathbf{o}, t_\mathbf{f}]$, where $\mathbf{x}^* \in c^1[t_\mathbf{o}, t_\mathbf{f}]$ is the optimal state trajectory, a special type of optimization problem where the decision variables are functions rather than real numbers, and $\lambda^* \in B \vee [t_\mathbf{o}, t_\mathbf{f}]$ is the optimal trajectory.

The Hamilton–Jacobi–Bellman (HJB) equation is a partial differential equation which is the pivot of the optimal control theory which clearly specifies the value function that gives the minimum cost or maximum reward for a given dynamical system with a corresponding cost or reward function. The HJB method is applicable in both deterministic and stochastic setting. The value function defines the value of the maximum reward or the minimum cost arising from the controlled process. The objective of the optimal control is to characterize the value function to obtain the control variable $u^*(t)$ which reward or cost approaches the optimum value $V(t, x) = J\big(x, u^*(t)\big)$ given some value of state controlled process x. When solved locally, the HJB assumes necessary condition for an optimum but when its solution is defined over the entire state space, the HJB equation becomes a necessary and sufficient condition for an optimum.

8.0 Brownian Motion Of Power Utiltiy Function

Utility function is a functional which describes the welfare of a consumer in all consumption mix representing both their welfare and preference. Power utility function is isoelastic and describes utility in relation to consumption and other economic variables concerning an investor. The power utility function is a particular type of hyperbolic absolute risk aversion (HARA) but remains the only class of utility functions which is characterized by constant relative risk aversion and accounts for the reason why it is defined as CRRA utility function mathematically expressed as $U(V) = \frac{V^{1-c}}{1-c}$, if $c \neq 1$, where c is the coefficient of relative risk aversion.

9.0 The Model Formulation And The Model

Assuming an investor trades two assets, a risky stock and a risk free bond which has a rate that is a linear function of time. The dynamics of the price of the free asset denoted by $B(t)$ is given by;

$$dB(t) = r(t)B(t)dt \quad (1)$$

dividing both sides of equation (1) by $B(t)$ we get

$$\frac{dB(t)}{B(t)} = r(t)dt\,. \quad (2)$$

And that of the risky asset described by the constant elasticity of variance model;

$$dS(t) = S(t)[\mu dt + bS^{\gamma}(t)dZ_1(t)], \quad (3)$$

for risky asset and

$$dS_0(t) = \mu S_0(t)dt,\ S_0(0) = 1, \text{ for riskless asset.}$$

Dividing both sides of equation (3) by $S(t)$ gives

$$\frac{dS(t)}{S(t)} = [\mu dt + bS^{\gamma}(t)dZ_1(t)] \quad (4)$$

where $S(t)$ denotes the risky asset price at time t, and μ, b are constants. μ is the appreciation rate of the risky asset as $\{Z(t): t > 0\}$ is a standard Brownian motion in a complete probability space $(\Omega, F, (F)_{t>0}, P)$. $(F)_{t>0}$ is the augmented filtration generated by the Brownian motion $Z(t)$. $bS^{\gamma}(t)$ is the instantaneous volatility and the elasticity γ, a parameter which satisfies the general condition $\gamma \leq 0$. If the elastic parameter $\gamma = 0$, then equation (4) the constant elasticity of variance CEV model reduces to a geometric Brownian motion.

The Ornstein-Uhlembeck process is one of several approaches used to model (with modifications) interest rates, currency, exchange rates and commodity prices stochastically. It is given as

$$dr(t) = \theta(\mu - r(t))dt + \sigma dZ_2(t) \qquad (5)$$

Let $\pi(t)$ be the amount of money the investor puts in the risky asset at time t, then $[V(t) - \pi(t)]$ is the money amount he invested in the risk free asset, where $V(t)$ is the total wealth investment in both assets.

Corresponding to the trading strategy $\pi(t)$, the dynamics of the wealth process is the stochastic differential equation (SDE)

$$dV(t) = \pi \frac{dS(t)}{S(t)} + [V(t) - \pi(t)] \frac{dB(t)}{B(t)} \ . \qquad (6)$$

Substituting (2) and (3) into equation (6) gives

$$dV(t) = \pi[\mu dt + bS^{\gamma}(t)dZ_1(t)] + [V(t) - \pi(t)]r(t)dt. \qquad (7)$$

This simplifies to

$$dV(t) = \{(\mu - r(t))\pi(t) + r(t)V(t)\}dt + bS^{\gamma}(t)\pi(t)dZ_1(t) \qquad (8)$$

The quadratic variation of equation using inner product definition (8)

implies that $< dV(t) > = [\{(\mu - r(t))\pi(t) + r(t)V(t)\}dt + bS^{\gamma}(t)\pi(t)dZ_1(t)]^2$

and hence

$$< dV(t) >= b^2 S^{2\gamma}(t)\pi^2(t)dt \qquad (9)$$

where

$$\left.\begin{aligned} dt.dt = dt.dZ_1(t) = 0 \\ dZ_1(t).dZ_1(t) = dt \end{aligned}\right\} \qquad (10)$$

The investor's problem is to find the optimal investment strategy for

$$G(V, t; T) = Max_{\pi(t)} E[U(V)] \qquad (11)$$

subject to:

$$dV(t) = \{(\mu - r(t))\pi(t) + r(t)V(t)\}dt + bS^{\gamma}(t)\pi(t)dZ_1(t).$$

10.0 The Optimization Problem

It is assumed that the investor has a power utility function $U(V)$ so that the optimization problem of the investor can be formulated as given above-find the optimal investment strategy for

$$G(V, t; T) = Max_{\pi(t)} E[U(V)]$$

subject to:

$$dV(t) = \{(\mu - r(t))\pi(t) + r(t)V(t)\}dt + bS^{\gamma}(t)\pi(t)dZ_1(t),$$

where $r(t)$ is the risk free rate, $\pi(t)$ is the amount invested in risky asset, $dZ_1(t)$ is the increase in the Wiener process, (μ, b) is the expected return and volatility of the stock market and v is the utility preference.

This study assumes that the investor has Power utility preference.

$$U(V) = \frac{-V^{1-c}}{1-c}; \ \ c \neq 1 \qquad (12)$$

which has a constant relative risk aversion c and coefficient of relative risk aversion defined as;

$$R(V) = -\frac{VU''(V)}{U'(V)} \qquad (13)$$

where V is the wealth level of the investor. As the relative risk aversion increases, the percentage invested in risky asset fall as amount of wealth increases, but as the relative risk aversion decreases, the percentage invested in risky asset rises as amount of wealth increases. The percentage invested in risky asset remains the same as amount of wealth increases as the relative risk aversion is constant. The utility function is unique up to a positive affine transform as the function $U''(V)$ only cannot be used to characterize the forcing intensity of the aversion coefficient which seems to have strictly positive affine transform invariance of utility preferences.

10.1 When the Brownian Motions Correlate.

That is $E[dz_1.dz_2] = \rho dt$ where $\boldsymbol{\rho}$ is the correlation function
Then, we have

$$\left.\begin{aligned} dV(t) &= \{(\mu - r(t))\pi(t) + r(t)V(t)\}dt + bS^{\gamma}(t)\pi(t)dZ_1(t) \\ dr(t) &= \theta(\mu - r(t))dt + \sigma dZ_2(t) \\ dS(t) &= \{S(t)[\mu dt + bS^{\gamma}(t)dZ_1]\} \\ < dS(t) > &= (dS)^2 = \{S(t)[\mu dt + bS^{\gamma}(t)dZ_1]\}^2 = b^2 S^{2(\gamma+1)}dt \\ < dr(t) > &= (dr)^2 = \sigma^2 dt \\ < dV(t) > &= (dV)^2 = b^2 S^{2\gamma}(t)\pi^2(t)dt = b^2 S^{2\gamma}\pi^2 dt \\ (dSdV) &= b^2 S^{2\gamma}(t)S(t)\pi(t)dt = b^2 S^{(2\gamma+1)}\pi dt \\ (drdV) &= \rho\sigma b S^{\gamma}\pi dt \\ (dSdr) &= \rho\sigma b S^{(\gamma+1)}dt \end{aligned}\right\} \quad (14)$$

where

$$\left.\begin{aligned} dt.dt = dt.dZ_1 = dt.dZ_2 &= 0 \\ dZ_1.dZ_1 = dZ_2.dZ_2 &= dt \\ dz_1.dz_2 &= \rho dt \end{aligned}\right\} \quad (15)$$

Again, substituting equation $dr(t) = \theta(\mu - r(t))dt + \sigma dZ_2(t)$ and (14) in maximum principle as stated below

$$dG = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial \mathbf{S}}dS + \frac{\partial G}{\partial \mathbf{r}}dr + \frac{\partial G}{\partial v}dV + \frac{\partial^2 G}{\partial \mathbf{S}\,\partial v}dSdV + \frac{\partial^2 G}{\partial \mathbf{r}\,\partial \mathbf{v}}drdV + \frac{\partial^2 G}{\partial \mathbf{S}\,\partial \mathbf{r}}dSdr + \frac{1}{2}\left[\frac{\partial^2 G}{\partial \mathbf{S}^2}(dS)^2 + \frac{\partial^2 G}{\partial r^2}(dr)^2 + \frac{\partial^2 G}{\partial \mathbf{v}^2}(dV)^2\right] \quad (16)$$

$$dG = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial S}[S(\mu dt + bS^{\gamma}dZ_1)] + \frac{\partial G}{\partial \mathbf{r}}[\alpha(\beta - r)dt + \sigma dZ_2] + \frac{\partial G}{\partial V}[\{(\mu - r)\pi + rV\}dt + bS^{\gamma}\pi dZ_1] + \frac{\partial^2 G}{\partial S\,\partial V}\left[b^2 S^{(2\gamma+1)}\pi dt\right] + \frac{\partial^2 G}{\partial \mathbf{r}\,\partial V}[\rho\sigma b S^{\gamma}\pi dt] + \frac{\partial^2 G}{\partial S\,\partial \mathbf{r}}\left[\rho\sigma b S^{(\gamma+1)}dt\right] + \frac{1}{2}\left[\frac{\partial^2 G}{\partial S^2}\left[b^2 S^{2(\gamma+1)}dt\right] + \frac{\partial^2 G}{\partial \mathbf{r}^2}[\sigma^2 dt] + \frac{\partial^2 G}{\partial V^2}[b^2 S^{2\gamma}\pi^2 dt]\right] \quad (17)$$

Using (17) in $\mathrm{Max}_{\boldsymbol{\pi}} \frac{1}{dt} E[dG] = 0$, we obtain;

$$\mathrm{Max}_{\boldsymbol{\pi}} \frac{1}{dt} E\left\{\frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial \mathbf{S}}[S(\mu dt + bS^{\gamma}dZ_1)] + \frac{\partial G}{\partial r}[\alpha(\beta - r)dt + \sigma dZ_2] + \frac{\partial G}{\partial v}[\{(\mu - r)\pi + rV\}dt + bS^{\gamma}\pi dZ_1] + \frac{\partial^2 G}{\partial \mathbf{S}\,\partial v}\left[b^2 S^{(2\gamma+1)}\pi dt\right] + \frac{\partial^2 G}{\partial \mathbf{r}\,\partial V}[\rho\sigma b S^{\gamma}\pi dt] + \frac{\partial^2 G}{\partial S\,\partial \mathbf{r}}\left[\rho\sigma b S^{(\gamma+1)}dt\right] + \frac{1}{2}\left[\frac{\partial^2 G}{\partial \mathbf{S}^2}\left[b^2 S^{2(\gamma+1)}dt\right] + \frac{\partial^2 G}{\partial \mathbf{r}^2}[\sigma^2 dt] + \frac{\partial^2 G}{\partial \mathbf{v}^2}[b^2 S^{2\gamma}\pi^2 dt]\right]\right\} = 0 \quad (18)$$

$$G_t + G_s\mu S + G_r[\alpha(\beta - r)] + G_v[(\mu - r)\pi + rV] + G_{sv}\left[b^2 S^{(2\gamma+1)}\pi\right] + G_{rv}[\rho\sigma b S^{\gamma}\pi] + G_{sr}\left[\rho\sigma b S^{(\gamma+1)}\right] + \frac{G_{ss}\left[b^2 S^{2(\gamma+1)}\right]}{2} + \frac{G_{rr}[\sigma^2]}{2} + \frac{G_{vv}\left[b^2 S^{2\gamma}\pi^2\right]}{2} = 0 \qquad (19)$$

where

$$E[dz_1] = [dz_2] = 0 \qquad (20)$$

Differentiating (19) with respect to π gives;

$$G_v(\mu - r) + G_{sv}\left(b^2 S^{(2\gamma+1)}\right) + G_{rv}(\rho\sigma b S^{\gamma}) + G_{vv}(b^2 S^{2\gamma}\pi) = 0 \qquad (21)$$

Making π the subject in (21), we get the optimal investment strategy as

$$\pi^* = -\left[\frac{(\mu-r)G_v}{(b^2 \mathbf{S}^{2\gamma})G_{vv}} + \frac{(\rho\sigma\mathbf{S}^{\gamma})G_{rv}}{(b^2\mathbf{S}^{2\gamma})G_{vv}} + \frac{\left(b^2 S^{(2\gamma+1)}\right)G_{sv}}{(b^2\mathbf{S}^{2\gamma})G_{vv}}\right]. \qquad (22)$$

In order to eliminate the dependency on V we let

$$G(t,s,r,V) = h(t,s,r)\frac{V^{1-c}}{1-c}, \qquad (23)$$

Such that the terminal time T

$$h(T,s,r) = 1, \qquad (24)$$

then we have from (23)

$$G_t = \frac{V^{1-c}}{1-c}h_t, G_s = \frac{V^{1-c}}{1-c}h_s, G_r = \frac{V^{1-c}}{1-c}h_r, G_v = V^{-c}h,\ G_{sv} = V^{-c}h_s,\ G_{rv} = V^{-c}h_r,\ G_{sr} = V^{-c}h_r,\ G_{ss} = \frac{V^{1-c}}{1-c}h_{ss},\ G_{rr} = \frac{V^{1-c}}{1-c}h_{rr},\ G_{vv} = -cV^{-(1+c)}h. \qquad (25)$$

Applying (25) in (22) and simplifying we get

$$\pi^* = \left[\frac{(\mu-r)V^{-c}h}{(b^2\mathbf{S}^{2\gamma})cV^{-(1+c)}h} + \frac{(\rho\sigma\mathbf{S}^{\gamma})V^{-c}h_r}{(b^2\mathbf{S}^{2\gamma})cV^{-(1+c)}h} + \frac{\left(b^2 S^{(2\gamma+1)}\right)V^{-c}h_s}{(b^2\mathbf{S}^{2\gamma})cV^{-(1+c)}h}\right], \qquad (26)$$

which becomes

$$\pi^* = \left[\frac{(\mu-r)}{c(b^2\mathbf{S}^{2\gamma})} + \frac{(\rho\sigma\mathbf{S}^{\gamma})h_r}{c(b^2\mathbf{S}^{2\gamma})h} + \frac{\left(b^2 S^{(2\gamma+1)}\right)Vh_s}{(b^2\mathbf{S}^{2\gamma})ch}\right]V. \qquad (27)$$

To eliminate the dependency on s, we let

$$h(t,r,s) = q(t,r)\frac{S^{1-c}}{1-c}, \qquad (28)$$

such that at terminal time T

$$q(T,r) = \frac{1-c}{S^{1-c}}. \qquad (29)$$

We get from (28) that

$$h_r = \frac{S^{1-c}}{1-c}q_r \text{ and } h_s = S^{-c}q. \qquad (30)$$

Using (28) and (30) in (27) we obtain the optimal investment strategy as

$$\pi^* = \left[\frac{(\mu-r)}{(cb^2\mathbf{S}^{2\gamma})} + \frac{\rho\sigma q_r}{(cb\mathbf{S}^{\gamma})q} + \frac{(1-c)}{c}\right]V. \qquad (31)$$

Further, to eliminate the dependency on r, we let

$$q(r,t) = \frac{r^{1-c}}{1-c}J(t) \qquad (32)$$

such that at terminal time T,

$$J(T) = \frac{(1-c)^2}{(r\mathbf{S})^{1-c}}. \qquad (33)$$

From (32), we get

$$q_r = r^{-c}J \qquad (34)$$

Substituting for $q(r,t)$ and q_r in (31) we use (32) and (34) respectively to get obtain

$$\pi^* = \left[\frac{(\mu-r)}{cb^2\mathbf{S}^{2\gamma}} + \frac{(1-c)\rho\sigma}{cbr\mathbf{S}^{\gamma}} + \frac{(1-c)}{c}\right]V. \qquad (35)$$

Equation (35) is the required investor's optimal investment strategy.

10.2 When the Brownian Motions Do Not Correlate

In this section we have that $E[dZ_1(t).dZ_1(t)] = 0$ and equations modify to (36) and (37) respectively as shown below.

$$dG = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial s}dS + \frac{\partial G}{\partial r}dr + \frac{\partial G}{\partial v}dV + \frac{\partial^2 G}{\partial s\,\partial v}dSdV + \frac{\partial^2 G}{\partial r\,\partial v}drdV + \frac{\partial^2 G}{\partial s\,\partial r}dSdr + \frac{1}{2}\left[\frac{\partial^2 G}{\partial s^2}(dS)^2 + \frac{\partial^2 G}{\partial r^2}(dr)^2 + \frac{\partial^2 G}{\partial v^2}(dV)^2\right] \quad (36)$$

$$\left.\begin{aligned} dV(t) &= \{(\mu - r(t))\pi(t) + r(t)V(t)\}dt + bS^{\gamma}(t)\pi(t)dZ_1(t) \\ dr(t) &= \theta(\mu - r(t))dt + \sigma dZ_2(t) \\ dS(t) &= \{S(t)[\mu dt + bS^{\gamma}(t)dZ_1]\} \\ < dS(t) >= (dS)^2 &= \{S(t)[\mu dt + bS^{\gamma}(t)dZ_1]\}^2 = b^2S^{2(\gamma+1)}dt \\ < dr(t) >= (dr)^2 &= \sigma^2 dt \\ < dV(t) >= (dV)^2 &= b^2S^{2\gamma}(t)\pi^2(t)dt = b^2S^{2\gamma}\pi^2 dt \\ (dSdV) &= b^2S^{2\gamma}(t)S(t)\pi(t)dt = b^2S^{(2\gamma+1)}\pi dt \\ (drdV) &= 0 \\ (dSdr) &= 0 \end{aligned}\right\} \quad (37)$$

where

$$\left.\begin{aligned} dt.dt = dt.dZ_1 = dt.dZ_2 &= 0 \\ dZ_1.dZ_1 = dZ_2.dZ_2 &= dt \\ dZ_1.dZ_2 &= 0 \end{aligned}\right\} \quad (38)$$

Similarly equation (17) becomes

$$dG = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial s}[S(\mu dt + bS^{\gamma}dZ_1)] + \frac{\partial G}{\partial r}[\alpha(\beta - r)dt + \sigma dZ_2] + \frac{\partial G}{\partial v}[\{(\mu - r)\pi + rV\}dt + bS^{\gamma}\pi dZ_1] + \frac{\partial^2 G}{\partial s\,\partial v}\left[b^2S^{(2\gamma+1)}\pi dt\right] + \frac{\partial^2 G}{\partial r\,\partial v}[0] + \frac{\partial^2 G}{\partial s\,\partial r}[0] + \frac{1}{2}\left[\frac{\partial^2 G}{\partial s^2}\left[b^2S^{2(\gamma+1)}dt\right] + \frac{\partial^2 G}{\partial r^2}[\sigma^2 dt] + \frac{\partial^2 G}{\partial v^2}[b^2S^{2\gamma}\pi^2 dt]\right]$$

$$dG = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial s}[S(\mu dt + bS^{\gamma}dZ_1)] + \frac{\partial G}{\partial r}[\alpha(\beta - r)dt + \sigma dZ_2] + \frac{\partial G}{\partial v}[\{(\mu - r)\pi + rV\}dt + bS^{\gamma}\pi dZ_1] + \frac{\partial^2 G}{\partial s\,\partial v}\left[b^2S^{(2\gamma+1)}\pi dt\right] + \frac{1}{2}\left[\frac{\partial^2 G}{\partial s^2}\left[b^2S^{2(\gamma+1)}dt\right] + \frac{\partial^2 G}{\partial r^2}[\sigma^2 dt] + \frac{\partial^2 G}{\partial v^2}[b^2S^{2\gamma}\pi^2 dt]\right] \quad (39)$$

and (19) becomes

$$\text{Max}_{\pi}\frac{1}{dt}E\left\{\frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial s}[S(\mu dt + bS^{\gamma}dZ_1)] + \frac{\partial G}{\partial r}[\alpha(\beta - r)dt + \sigma dZ_2] + \frac{\partial G}{\partial v}[\{(\mu - r)\pi + rV\}dt + bS^{\gamma}\pi dZ_1] + \frac{\partial^2 G}{\partial S\,\partial v}\left[b^2S^{(2\gamma+1)}\pi dt\right] + \frac{1}{2}\left[\frac{\partial^2 G}{\partial S^2}\left[b^2S^{2(\gamma+1)}dt\right] + \frac{\partial^2 G}{\partial r^2}[\sigma^2 dt] + \frac{\partial^2 G}{\partial v^2}[b^2S^{2\gamma}\pi^2 dt]\right]\right\} = 0$$

(40)

Fro (40) we obtain

$$G_t + G_s\mu S + G_r[\alpha(\beta - r)] + G_v[(\mu - r)\pi + rV] + G_{sv}\left[b^2S^{(2\gamma+1)}\pi\right] + \frac{G_{ss}\left[b^2S^{2(\gamma+1)}\right]}{2} + \frac{G_{rr}[\sigma^2]}{2} + \frac{1}{2}G_{vv}[b^2S^{2\gamma}\pi^2] = 0. \quad (41)$$

Differentiating (41) with respect to π we get

$$G_{v}(\mu - r) + G_{sv}\big(b^2 S^{(2\gamma+1)}\big) + G_{vv}(b^2 S^{2\gamma}\pi) = 0, \qquad (42)$$

From which we have

$$\pi^* = -\left[\frac{(\mu - r)G_v}{(b^2 S^{2\gamma})G_{vv}} + \frac{(b^2 S^{(2\gamma+1)})G_{sv}}{(b^2 S^{2\gamma})G_{vv}}\right] \qquad (43)$$

Applying (24) and (25), replacing for G_{sv}, G_v, and G_{vv} in (43) we

$$\pi^* = \left[\frac{(\mu - r)V}{cb^2 S^{2\gamma}} + \frac{SVh_s}{ch}\right]. \qquad (44)$$

Going through steps (28) to (35) ,we obtain the optimal investment strategy

$$\pi^* = \left[\frac{(\mu - r)}{cb^2 S^{2\gamma}} + \frac{(1-c)}{c}\right]V \ . \qquad (45)$$

Since c is the aversion co-efficient, it is possible to assume 0.5 as a value for c so that

$$\pi^* = \frac{2(\mu - r)}{b^2 S^{2\gamma}}V + V, \qquad (46)$$

from which

$$\pi^* - V = \frac{2(\mu - r)}{b^2 S^{2\gamma}}. \qquad (47)$$

11 Discussions of Results: Four Implications of Correlation of the Brownian Motions

The optimal investment strategy when the Brownian motions do not correlate is given by

$$\pi^*_{nc} = \left[\frac{(\mu - r)}{cb^2 S^{2\gamma}} + \frac{(1-c)}{c}\right]V$$

and when the Brownian motions correlate by;

$$\pi^*_{c} = \left[\frac{(\mu - r)}{cb^2 S^{2\gamma}} + \frac{(1-c)\rho\sigma}{cbrS^{\gamma}} + \frac{(1-c)}{c}\right]V,$$

therefore

$$\pi^*_{c} = \left[\frac{(\mu - r)}{cb^2 S^{2\gamma}} + \frac{(1-c)\rho\sigma}{cbrS^{\gamma}} + \frac{(1-c)}{c}\right]V,$$

and

$$\pi^*_{c} = \pi^*_{nc} + \left[\frac{(1-c)\rho\sigma}{cbrS^{\gamma}}\right]V. \qquad (48)$$

We look into the following four cases for the correlation of the Brownian motions namely:

(i). when the correlation is said to be unity $\rho = 1$, that is;

$$\rho = 1, \qquad (49)$$

substituting equation (49) into (48) we obtain

$$\pi^*_{c} = \pi^*_{nc} + \left[\frac{(1-c)\sigma}{cbrS^{\gamma}}\right]V, \qquad (50)$$

that is, when $\rho = 1$, the investor optimal investment strategy when the Brownian motions correlate is greater than the optimal investment strategy when the Brownian motions do not correlate by a fraction $\left[\frac{(1-c)\sigma}{cbrS^{\gamma}}\right]$ of the total wealth.

(ii). when the correlation is said to be negative say $\rho = -k$, that is;

$$\rho = -k \qquad (51)$$

Substituting equation (51) into (48) we obtain

$$\pi^*_{c} = \pi^*_{nc} - \left[\frac{k(1-c)\sigma}{cbrS^{\gamma}}\right]V. \qquad (52)$$

This implies that when $\rho = -k$ the investor's optimal investment strategy when the Brownian motions correlate is less than the optimal investment strategy when the Brownian motions do not correlate by a fraction $\left[\frac{k(1-c)\sigma}{cbrS^{\gamma}}\right]$ of the total wealth.

(iii): when the correlation is said to be positive say $\rho = k$

$$\pi_c^* = \pi_{nc}^* + \left[\frac{k(1-c)\sigma}{cbr\mathbf{S}^\gamma}\right]V,$$

the investor's optimal investment strategy when Brownian motions correlate is greater than the investor optimal investment strategy when the Brownian motions do not correlate by a fraction $\left[\frac{k(1-c)\sigma}{cbr\mathbf{S}^\gamma}\right]$ of the total wealth.

(iv): when the correlation is equal zero say $\rho = 0$, $\pi_c^* = \pi_{nc}^*$ the investor's optimal investment strategy when Brownian motions correlate is equal to the investor optimal investment strategy when Brownian motions do not correlate.

Finally, when $\gamma = 0$, $(\pi_c^* - \pi_{nc}^*) = \left[\frac{(1-c)\rho\sigma}{cbr}\right]V$, there will be leverage effect condition in equity market where the volatility of a stock increases as its price falls. In particular, $\gamma = \frac{1}{2}$, and $\pi_c^* = \pi_{nc}^* - \left[k(1-c)\sigma\, cbrS^{-\frac{1}{2}}\right]V$, we arrive at the square root process as observed in Heston model.

When $\gamma = 1$, $(\pi_c^* - \pi_{nc}^*) = \left[\frac{(1-c)\rho\sigma}{cbr\mathbf{S}}\right]V$ then the model reduces to geometric Brownian motion type.

It is generally observed given that $0 < \zeta < 1$ the distribution of μ skews to the left t, however the distribution will skew to the right whenever $\gamma > 0$. An advantage of CVE model is that it does not generate extra risk or randomness but rather preserves market completeness while maintaining a tractable stochastic form associating smoothly to geometric Brownian motion models.

12.0 Conclusion

Within the stochastic optimal control framework, the optimal investment strategy problem was formulated. The Ito's Lema in tandem with laws of stochastic calculus was applied and we obtained the HJB equation for the optimal control problem. Consequently, a closed form stochastic solution for the optimal control policies and the associating value functions under power utility preference was provided. This article offers a good description of optimal stochastic optimal investment strategy and a comparison is made concerning stochastic interest rate and correlating Brownian motions. The solution $\pi_c^* = \pi_{nc}^* + \left[\frac{(1-c)\rho\sigma}{cbr\mathbf{S}^\gamma}\right]V$ clearly shows that the value of the investor optimal investment strategy when the Brownian motions correlate is greater than the investor optimal investment strategy when the Brownian motions do not correlate by a fraction $\left[\frac{(1-c)\rho\sigma}{cbr\mathbf{S}^\gamma}\right]$ of the total wealth. We have evaluated the implications of the correlation under four scenerios when the correlation is unity, greater than zero, less than zero and when the correlation is $\mathbf{o}(1)$ where $\mathbf{o}(1)$ is a function which vanishes. If the correlation is unity, then the investor's optimal investment strategy when the Brownian motions correlate is greater than the investor optimal investment strategy when the Brownian motions do not correlate by a fraction $\left[\frac{(1-c)\rho\sigma}{cbr\mathbf{S}^\gamma}\right]$ of the total wealth**.** When the correlation is positive, the investor optimal investment strategy when the Brownian motions correlate is greater than the investor optimal investment strategy when the Brownian motions do not correlate by a fraction $\Psi\left[\frac{(1-c)\sigma}{cbr\mathbf{S}^\gamma}\right]$ of the total wealth, where Ψ is a positive constant. When the correlation is negative, the investor optimal investment strategy when the Brownian motions correlate is less than the investor optimal investment strategy when the Brownian motions do not correlate by just a fraction $\left[\frac{k(1-c)\sigma}{cbr\mathbf{S}^{\gamma}}\right]$ of the wealth where k is a constant. When the correlation equals zero, then the investor optimal investment strategy when the Brownian motions correlate is equal to the investor optimal investment strategy when the Brownian motions do not correlate.

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