Physics of fluid motion

1

O.C. Okoye, B.O. Bolaji
Department of Mechanical Engineering, Federal University, Oye Ekiti, Ikole Ekiti Campus, Ekiti State, Nigeria

1.1 Introduction

Fluids are made up of molecules that are continuously in motion. Most engineering applications deal with the average or macroscopic effects of several fluid molecules. A fluid is regarded as a substance which is infinitely divisible, that is, a continuum, and the focus are not on the behavior of the respective molecules of the fluid. The continuum is a theoretical continuous medium used to replace the molecular structure, when dealing with the relationships of fluid flow on an analytical or mathematical basis [14]. The continuum hypothesis treats the fluid as being infinitely divisible without character change. Therefore, material properties like viscosity, density or thermal conductivity, and variables such as pressure, temperature and velocity, can be defined at a mathematical point as the limit of the mean of the given variable or quantity across the molecular fluctuations [22].

The method of continuum mechanics is very useful in offering physical explanation and mathematical description of different transport phenomena without the need to fully understand the internal micro- and nanostructures of fluids [20]. Classical fluid mechanics is based on the concept of a continuum. When considering the behavior of fluids under normal conditions, the continuum assumption holds true. In situations in which the mean free path of the molecules is of the same order of magnitude as the least significant characteristic dimension, the continuum assumption becomes invalid. In such situations, for example, rarefied gas flow, the concept of a continuum is discarded and the microscopic and statistical viewpoints are adopted [6,32,40]. Solid structures, as well as fluid flow fields [20], are assumed to be continua insofar as the local material properties are defined as averages across material elements or volumes significantly larger than the microscopic length scales of the solid or fluid while being small in comparison with the macroscopic structure. Mathematically, for the continuum hypothesis to be valid, the following condition must be satisfied [12,22]:

$$\boldsymbol{d} \ll V^{1/3} \ll L,$$

where

$\boldsymbol{d}$ is the length scale which is representative of the microstructure of the fluid, most often a molecular length scale,
$V^{1/3}$ is the characteristic linear dimension of the averaging volume,
L is the macroscopic scale which is typical for spatial gradients in the variables that were averaged. The size of the flow domain usually dictates the scale of L.

Adopting the continuum hypothesis, the macroscopic laws of classical continuum physics are used to give a mathematical description of fluid motion, as well as heat transfer in systems that are not isothermal. These laws are: the law of conservation of mass, the law of conservation of linear and angular momentum, and the law of conservation of energy (i.e., the first law of thermodynamics). While the second law of thermodynamics is not directly used in the derivation of the governing equations, it provides constraints on the permissible forms of the constitutive models which relate the gradients of velocity in the fluid to short-range forces acting across the surfaces inside the fluid [22,23].

Most theoretical studies in fluid dynamics are based on the concept of a perfect fluid. A perfect fluid is one that is both frictionless and incompressible. When perfect fluids move, two fluid layers in contact with each other do not experience tangential forces (i.e., shearing stresses). However, they act on each other with normal forces (i.e., pressure) only. Thus, the perfect fluid has no internal resistance to change the shape of the fluid. The mathematical theory of the motion of a perfect fluid is very well developed, and gives a description of real fluid motion which is satisfactory in many situations, for instance, the formation of jets of liquid in air. However, the theory of perfect fluids cannot explain the drag of a body. In real fluids, the inner layers of the fluid transmit both tangential and normal stresses, and this is also true near a solid wall which is in contact with the fluid. Thus, the results obtained from the theory of a perfect fluid are not acceptable in such situations. The tangential or friction forces in a real fluid are related to the viscosity of the fluid [16,38].

The viscosity of real fluids can be considerably affected by shear rate, temperature, pressure, molecular structure, molecular weight, and time of shearing. The viscosity of a gas increases with temperature, but the viscosity of a liquid decreases with temperature. There are no inviscid fluids in reality [6,28,29]. Nevertheless, in some situations, the effects of viscosity are relatively small when compared with other effects. In such situations, therefore, viscous effects can often be neglected. For instance, viscous forces which are developed in flowing water may be several orders of magnitude less than the forces as a result of other influences like gravity or pressure differences [28].

There are two ways of analyzing the motion of a mass of fluid upon which forces act in certain conditions. The equations of the motion of a fluid have been obtained based on any of the given methods of analysis. One method of analysis is to track the history of all particles of the fluid, while the other method investigates the velocity, density, and pressure at every point of space which the fluid occupies at all points in time. These obtained equations are respectively known as the Lagrangian and Eulerian forms of the hydrokinetic equations [44].

1.2 The basic equations of viscous flow

Viscous flow equations have been available for over a century. The complete form of these equations is impossible to solve even with the high computing power of modern digital computers. The equations for turbulent flow cannot be solved with current

mathematical techniques since the boundary conditions are randomly dependent on time [5,43]. Derivations of the equations of fluid dynamics [20] give a more in-depth understanding of the physics which all of the terms in the final equations mathematically represent. It also provides insight into the assumptions (or the shortcomings) of a given mathematical model [1].

White [43] noted that the exact number of fundamental equations of compressible viscous flow is a matter of personal choice, as some relations are more fundamental than others. In this chapter, the system is considered to have only three fundamental relations along with four auxiliary relations. The fluid is assumed to be uniform and homogeneous in composition. The fundamental equations are the three laws of conservation for physical systems, namely:

- The law of conservation of mass (continuity)
- The law of conservation of momentum (Newton's second law)
- The law of conservation of energy (the first law of thermodynamics).

These three fundamental equations are used to obtain three unknowns, which are [33]:

- The thermodynamic pressure, p
- The absolute temperature, T, and
- The velocity vector, V.

Assuming local thermodynamic equilibrium, p and T are considered to be the two independent thermodynamic variables needed. In the final forms of the conservation equations, four additional thermodynamic variables are included. They are the enthalpy (or internal energy), density, thermal conductivity, and dynamic viscosity. The thermodynamic pressure, p, and the absolute temperature, T, can, therefore, be used to uniquely obtain these additional four variables. The basic equations which will be derived are largely general and based on the following assumptions [43]:

- The fluid constitutes a (mathematical) continuum.
- The particles of the fluid are basically in thermodynamic equilibrium.
- The conduction of heat is according to Fourier's law.
- Body forces are only a result of gravity.
- There are no internal sources of heat.

1.2.1 Law of conservation of mass: the continuity equation

The law of conservation of mass relates the density field to the velocity field. An infinitesimal control volume will be used to derive the law of conservation of mass in both the rectangular and cylindrical coordinate systems [24].

1.2.2 Derivation of the law of conservation of mass in the rectangular coordinate system

Consider an infinitesimal cube having dimensions dx, dy, and dz (see Fig. 1.1). At the center of the cube, O, the density is ρ, and the velocity has components u, v, and w,
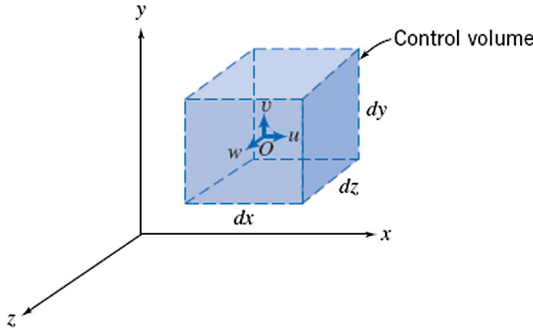
Figure 1.1 Differential control volume in the rectangular coordinate system [35].

in the x, y, and z axes, respectively. By using the truncated Taylor series expansion of the density multiplied by the normal component of velocity at the midpoint of all the six faces of the cube gives [4]:

Center of front face: $(\rho w) \cong \rho w + \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}$,

Center of rear face: $(\rho w) \cong \rho w - \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}$,

Center of right face: $(\rho u) \cong \rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}$,

Center of left face: $(\rho u) \cong \rho u - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}$,

Center of top face: $(\rho v) \cong \rho v + \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}$,

Center of bottom face: $(\rho v) \cong \rho v - \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}$.

The mass flow rates through each face of the cube are shown in Fig. 1.2. They were obtained by calculating the product of the density, the surface area of each face, and the normal component of velocity at the midpoint of each face. The rate of change of mass of the control volume as it shrinks to a point is [4]:

$$\int \frac{\partial \rho}{\partial t} dv \cong \frac{\partial \rho}{\partial t}\, dxdydz$$

where $dxdydz$ = volume of the cube. Substituting the appropriate terms into the integral form of the continuity equation and simplifying gives [6,28,34]:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0. \tag{1.1}$$

Eq. (1.1) is the differential form of the conservation of mass equation in the rectangular coordinate system. It is also referred to as the continuity equation. Eq. (1.1) is valid for both steady and unsteady flow, as well as for both incompressible and compressible fluids [28]. In vector notation, Eq. (1.1) is written as [4,7,28]:

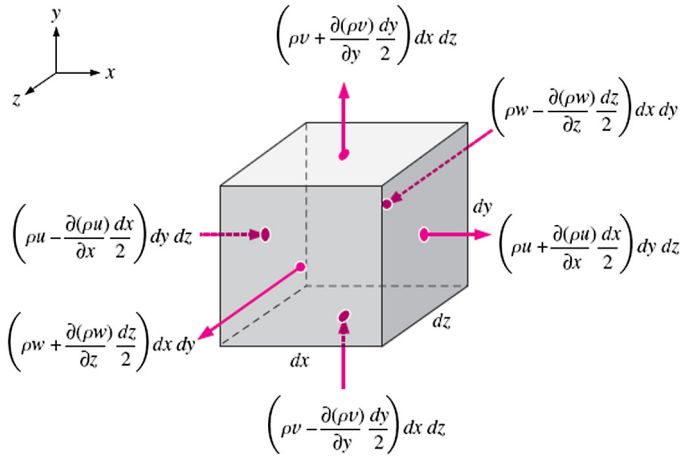$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho V = 0. \tag{1.2}$$

Figure 1.2 Mass flow rate through each face of the control volume [4].

(A)   (B)
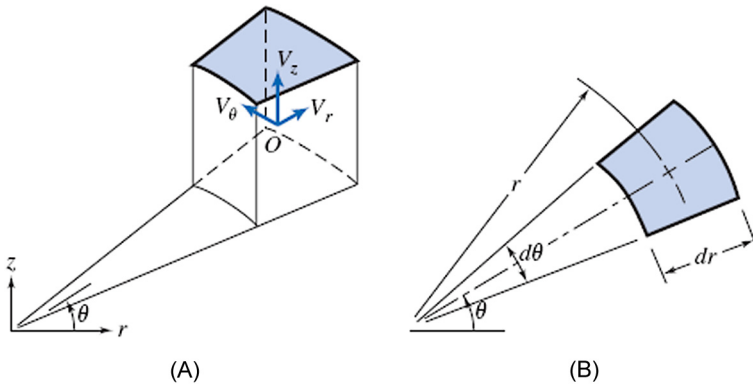
Figure 1.3 Differential control volume in cylindrical coordinate system [35]. (A) Isometric view; (B) Projection on $r\theta$ plane.

1.2.3   Derivation of the law of conservation of mass in the cylindrical coordinate system

The differential control volume for the cylindrical coordinate system is shown in Fig. 1.3. At the center, O, of the control volume, the density is ρ and the components of the velocity in the r, θ, and z directions are V_r, V_θ, and V_z, respectively. Using the truncated Taylor series expansion, the mass flux through the six faces of the control volume is given in Table 1.1. The components of velocity in the r, θ, and z directions are assumed to be in the positive direction.

Thus, the net rate of mass flux going out through the control surface is [26,41]:

$$\left[\rho V_r + r\frac{\partial \rho V_r}{\partial r} + \frac{\partial \rho V_\theta}{\partial \theta} + r\frac{\partial \rho V_z}{\partial z}\right] dr\, d\theta\, dz.$$

Since the volume of the fluid element is "$r d\theta dr dz$" and the density within the control volume is ρ, the rate at which mass changes within the control volume is:

$$\frac{\partial \rho}{\partial t} r d\theta dr dz.$$

The differential equation for the conservation of mass in the cylindrical coordinate system becomes

$$\rho V_r + r \frac{\partial \rho V_r}{\partial r} + \frac{\partial \rho V_\theta}{\partial \theta} + r \frac{\partial \rho V_z}{\partial z} + r \frac{\partial \rho}{\partial t} = 0,$$

or

$$\frac{\partial (r \rho V_r)}{\partial r} + \frac{\partial \rho V_\theta}{\partial \theta} + r \frac{\partial \rho V_z}{\partial z} + r \frac{\partial \rho}{\partial t} = 0.$$

By dividing through by r, the equation becomes

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial (r \rho V_r)}{\partial r} + \frac{1}{r} \frac{\partial (\rho V_\theta)}{\partial \theta} + r \frac{\partial (\rho V_z)}{\partial z} = 0. \tag{1.3}$$

In vector notation, Eq. (1.3) is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho V = 0.$$

1.3　Momentum equation

By applying Newton's second law to an infinitesimal particle of fluid having mass dm, the differential form of the momentum equation can be derived. Newton's second law for a finite system is

$$F = \frac{d P}{dt},$$

where F is the net force acting on the system and P is the linear momentum of the system. Rewriting Newton's second law for an infinitesimal system having mass dm gives [6]

$$d F = dm \frac{d V}{dt}.$$

Since the acceleration of an element of fluid having mass dm, which moves in a velocity field, is given by

$$a = \frac{D V}{Dt} = \frac{\partial V}{\partial t} + u \frac{\partial V}{\partial x} + v \frac{\partial V}{\partial y} + w \frac{\partial V}{\partial z},$$

Table 1.1 Mass flux through the control surface of a cylindrical differential control volume [6].

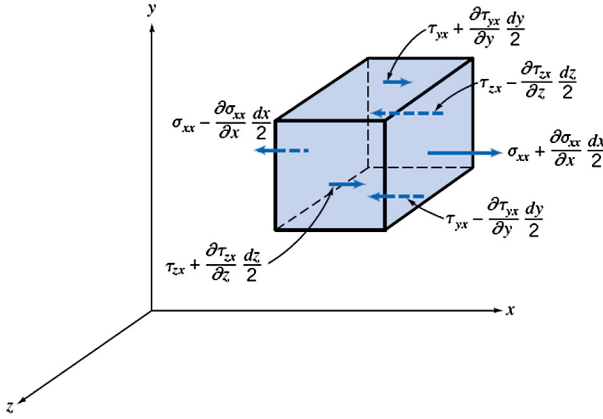| Surface	$\int \rho V \cdot dA$
Inside	$(-r) = -\left[\rho - \left(\frac{\partial \rho}{\partial r}\right)\frac{dr}{2}\right]\left[V_r - \left(\frac{\partial V_r}{\partial r}\right)\frac{dr}{2}\right]\left(r - \frac{dr}{2}\right)d\theta dz = -\rho V_r r d\theta dz + \rho V_r \frac{dr}{2}d\theta dz + \rho \left(\frac{\partial V_r}{\partial r}\right)r\frac{dr}{2}d\theta dz + V_r \left(\frac{\partial \rho}{\partial r}\right)r\frac{dr}{2}d\theta dz$
Outside	$(+r) = \left[\rho + \left(\frac{\partial \rho}{\partial r}\right)\frac{dr}{2}\right]\left[V_r + \left(\frac{\partial V_r}{\partial r}\right)\frac{dr}{2}\right]\left(r + \frac{dr}{2}\right)d\theta dz = \rho V_r r d\theta dz + \rho V_r \frac{dr}{2}d\theta dz + \rho \left(\frac{\partial V_r}{\partial r}\right)r\frac{dr}{2}d\theta dz + V_r \left(\frac{\partial \rho}{\partial r}\right)r\frac{dr}{2}d\theta dz$
Front	$(-\theta) = -\left[\rho - \left(\frac{\partial \rho}{\partial \theta}\right)\frac{d\theta}{2}\right]\left[V_\theta - \left(\frac{\partial V_\theta}{\partial \theta}\right)\frac{d\theta}{2}\right]dr dz = -\rho V_\theta dr dz + \rho \left(\frac{\partial V_\theta}{\partial \theta}\right)\frac{d\theta}{2}dr dz + V_\theta \left(\frac{\partial \rho}{\partial \theta}\right)\frac{d\theta}{2}dr dz$
Back	$(+\theta) = \left[\rho + \left(\frac{\partial \rho}{\partial \theta}\right)\frac{d\theta}{2}\right]\left[V_\theta + \left(\frac{\partial V_\theta}{\partial \theta}\right)\frac{d\theta}{2}\right]dr dz = \rho V_\theta dr dz + \rho \left(\frac{\partial V_\theta}{\partial \theta}\right)\frac{d\theta}{2}dr dz + V_\theta \left(\frac{\partial \rho}{\partial \theta}\right)\frac{d\theta}{2}dr dz$
Bottom	$(-z) = -\left[\rho - \left(\frac{\partial \rho}{\partial z}\right)\frac{dz}{2}\right]\left[V_z - \left(\frac{\partial V_z}{\partial z}\right)\frac{dz}{2}\right]r d\theta dr = -\rho V_z r d\theta dr + \rho \left(\frac{\partial V_z}{\partial z}\right)\frac{dz}{2}r d\theta dr + V_z \left(\frac{\partial \rho}{\partial z}\right)\frac{dz}{2}r d\theta dr$
Top	$(+z) = \left[\rho + \left(\frac{\partial \rho}{\partial z}\right)\frac{dz}{2}\right]\left[V_z + \left(\frac{\partial V_z}{\partial z}\right)\frac{dz}{2}\right]r d\theta dr = \rho V_z r d\theta dr + \rho \left(\frac{\partial V_z}{\partial z}\right)\frac{dz}{2}r d\theta dr + V_z \left(\frac{\partial \rho}{\partial z}\right)\frac{dz}{2}r d\theta dr$
Then,	$\int \rho V \cdot dA = \left[\rho V_r + r\left\{\rho\left(\frac{\partial V_r}{\partial r}\right) + V_r\left(\frac{\partial \rho}{\partial r}\right)\right\} + \left\{\rho\left(\frac{\partial V_\theta}{\partial \theta}\right) + V_\theta\left(\frac{\partial \rho}{\partial \theta}\right)\right\} + r\left\{\rho\left(\frac{\partial V_z}{\partial z}\right) + V_z\left(\frac{\partial \rho}{\partial z}\right)\right\}\right]dr d\theta dz$
or	$\int \rho V \cdot dA = \left[\rho V_r + r\frac{\partial \rho V_r}{\partial r} + \frac{\partial \rho V_\theta}{\partial \theta} + \frac{\partial \rho V_z}{\partial z}\right]dr d\theta dz$

Figure 1.4 Stresses in the x direction on a fluid element [35].

Newton's second law can be written as [6]

$$dF = dm\frac{DV}{Dt} = dm\left[\frac{\partial V}{\partial t} + u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y} + w\frac{\partial V}{\partial z}\right].\tag{1.4}$$

In order to obtain expressions for the forces acting on the fluid element in the three coordinate directions, consider the element of fluid shown in Fig. 1.4. In the figure only the x component of the stresses which result in the x component of the surface forces are shown. The fluid element is of mass dm and volume $dxdydz$. The stresses at the center of the fluid element are σ_{xx}, τ_{yx}, τ_{zx}, and the stresses (obtained using the truncated Taylor's series expansion about the fluid element's center) which act on all the six faces of the fluid element in the x direction are shown in Fig. 1.4. The net surface force in the x direction is obtained by summing the forces in that direction:

$$\begin{aligned}
dF_{S_x} &= \left(\sigma_{xx} + \frac{\partial\sigma_{xx}}{\partial x}\frac{dx}{2}\right)dydz - \left(\sigma_{xx} - \frac{\partial\sigma_{xx}}{\partial x}\frac{dx}{2}\right)dydz \\
&+ \left(\tau_{yx} + \frac{\partial\tau_{yx}}{\partial y}\frac{dy}{2}\right)dxdz - \left(\tau_{yx} - \frac{\partial\tau_{yx}}{\partial y}\frac{dy}{2}\right)dxdz \\
&+ \left(\tau_{zx} + \frac{\partial\tau_{zx}}{\partial z}\frac{dz}{2}\right)dxdy - \left(\tau_{zx} - \frac{\partial\tau_{zx}}{\partial z}\frac{dz}{2}\right)dxdy.
\end{aligned}$$

Simplifying this equation gives

$$dF_{S_x} = \left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z}\right)dxdydz.$$

If the force due to gravity is assumed to be the only body force acting on the fluid element, then the net force in the x direction, dF_x, is as presented below. The expressions for the force components in the y and z directions are derived in a similar way,

and are presented as:

$$dF_x = dF_{B_x} + dF_{S_x} = \left(\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) dxdydz, \qquad (1.5a)$$

$$dF_y = dF_{B_y} + dF_{S_x} = \left(\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) dxdydz, \qquad (1.5b)$$

$$dF_z = dF_{B_z} + dF_{S_z} = \left(\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \right) dxdydz. \qquad (1.5c)$$

Substituting the expressions for the components of the force in the x, y, and z directions into Eq. (1.4) gives the general differential equations of motion for fluids. The equations are valid for any fluid which satisfies the continuum assumption. These equations are presented as [6,28]:

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right), \qquad (1.6a)$$

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right), \qquad (1.6b)$$

$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right). \qquad (1.6c)$$

1.3.1 Constitutive relations for the equation of motion for Newtonian fluids

The constitutive relations applicable to Newtonian fluids are [8]:

$$\sigma_{ii} = -P + 2\mu \frac{\partial u_i}{\partial x_i} + l\nabla \cdot V, \qquad (1.7)$$

$$\tau_{ij} = \tau_{ji} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \qquad (1.8)$$

where i and j are the indices that represent the components of the Cartesian coordinates. For Newtonian fluids, the shear stresses are proposed to be directly proportional to the time rate of deformation of an element of a fluid with the viscosity coefficient m being the factor of proportionality [19]. From Stoke's assumption, the coefficient of bulk viscosity, l, is given by [8,9,43]

$$l = -\frac{2}{3}\mu. \qquad (1.9)$$

Thus, in the rectangular coordinate system [19] and [8]:

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right), \tag{1.10a}$$

$$\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right), \tag{1.10b}$$

$$\tau_{zx} = \tau_{xz} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), \tag{1.10c}$$

$$\sigma_{xx} = -p - \frac{2}{3}\mu \nabla \cdot V + 2\mu \frac{\partial u}{\partial x}, \tag{1.11a}$$

$$\sigma_{yy} = -p - \frac{2}{3}\mu \nabla \cdot V + 2\mu \frac{\partial v}{\partial y}, \tag{1.11b}$$

$$\sigma_{zz} = -p - \frac{2}{3}\mu \nabla \cdot V + 2\mu \frac{\partial w}{\partial z}, \tag{1.11c}$$

where p is the local thermodynamic pressure. The thermodynamic pressure is linked to the temperature and density by the equation of state [6]. In most applications, p is the only variable of importance. For a fluid with constant density, the second term in the normal stress equations is always zero. Only in cases of very large gradients of velocity in the orientation of the stress, in the last term of the normal stress equations is p significantly different from σ. In the analysis of a normal shock wave, for instance, all three terms in the shear stress equations are significant [8,19]. In concise form, the Cartesian stress tensor elements can be written as

$$\sigma_{ij} = - \left(p + \frac{2}{3}\mu \frac{\partial u_k}{\partial x_k} \right) \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

The Newtonian fluid stress tensor elements in cylindrical coordinates are presented below:

$$\tau_{rr} = \mu \left(2\frac{\partial u_r}{\partial r} \right) + l \nabla \cdot V, \tag{1.12}$$

$$\tau_{\theta\theta} = \left[2 \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) \right] + l \nabla \cdot V, \tag{1.13}$$

$$\tau_{zz} = \mu \left(2\frac{\partial u_z}{\partial z} \right) + l \nabla \cdot V, \tag{1.14}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu \left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r} \right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta} \right], \tag{1.15}$$

$$\tau_{\theta z} = \tau_{z\theta} = \mu \left[\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right], \tag{1.16}$$

$$\tau_{zr} = \tau_{rz} = \mu \left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right]. \tag{1.17}$$

1.3.2 Equations of motion for Newtonian fluids: Navier–Stokes equations

Substituting the expressions for the stresses into the differential equations of motion (i.e., Eq. (1.4)) gives the Navier–Stokes equations presented below [8,30]:

$$\rho \frac{Du}{Dt} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left[\mu \left(2\frac{\partial u}{\partial x} - \frac{2}{3} \nabla \cdot V \right) \right] + \frac{\partial}{\partial y} \left[\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right]$$
$$+ \frac{\partial}{\partial z} \left[\mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right],$$

$$\rho \frac{Dv}{Dt} = \rho g_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x} \left[\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + \frac{\partial}{\partial y} \left[\mu \left(2\frac{\partial v}{\partial y} - \frac{2}{3} \nabla \cdot V \right) \right]$$
$$+ \frac{\partial}{\partial z} \left[\mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right],$$

$$\rho \frac{Dw}{Dt} = \rho g_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial x} \left[\mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] + \frac{\partial}{\partial y} \left[\mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right]$$
$$+ \frac{\partial}{\partial z} \left[\mu \left(2\frac{\partial w}{\partial z} - \frac{2}{3} \nabla \cdot V \right) \right].$$

For incompressible flow having constant viscosity, the Navier–Stokes equations become [4,9,27,28]:

$$\rho \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right) = \rho g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \tag{1.18a}$$

$$\rho \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right) = \rho g_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right), \tag{1.18b}$$

$$\rho \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right) = \rho g_z - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right).$$
$$\tag{1.18c}$$

The Navier–Stokes equations for constant viscosity and density are presented below in the cylindrical coordinate system [9,17,18]:

(*r* component)

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z} \right)$$

$$= \rho g_r - \frac{\partial p}{\partial r} + \mu \left\{ \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} [r v_r] \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right\}, \quad (1.19)$$

(θ component)

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right)$$

$$= \rho g_\theta - \frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left\{ \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} [r v_\theta] \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right\},$$

$$(1.20)$$

(z component)

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right)$$

$$= \rho g_z - \frac{\partial p}{\partial z} + \mu \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right\}. \quad (1.21)$$

1.3.3   Constitutive equations for non-Newtonian fluids

Non-Newtonian fluids like large molecular weight polymers which do not follow the constitutive equations for Newtonian fluids are common in the chemical and plastics industry. Substances such as paints, toothpastes, and lubricants show non-Newtonian behavior. Such fluids display effects like climbing of a rod which rotates in a stationary fluid container, turning into semisolid under the application of an electric or magnetic field and die swell when leaving a tube [24].

As [10] noted, with the continuous increase in the use of plastics in the modern society, being able to predict their behavior is of immense economic value in manufacturing processes. The manner in which non-Newtonian viscosity varies with shear rate is remarkable, and cannot be neglected in modeling stress which acts when molten polymers and many biomedical and industrial materials flow. In such flow systems, the viscosity changes in a very considerable manner. The stress–velocity relationship for Newtonian fluids is not valid for such systems, since the viscosity changes. Therefore, these systems need a different non-Newtonian constitutive equation [27]. Various theoretical models have been proposed within the past century; however, prediction of the flow behavior of non-Newtonian fluids is not adequate. In practice, the parameters of a particular constitutive model are obtained from conducting some simple experiments. More experiments are then used to validate the predictions of the given model on other flow geometries. In most cases, the predictions are correct for a handful of simple flows with character very similar to those from which the variables of the given constitutive model were obtained. More studies are still required about the behavior of non-Newtonian fluids [13].

Non-Newtonian fluids display certain characteristics like normal stress effects and shear thinning, so that one single constitutive relation is not adequate for the description of the various phenomena [39]. Morrison [27] noted that the study of non-

Newtonian effects is a complete branch of knowledge on its own, and a considerable number of constitutive models have been proposed for non-Newtonian fluids. Morrison [27] discusses only two classes of non-Newtonian constitutive equations, namely, inelastic and viscoelastic. Most of the nomenclature employed in the field is based on simple experiments in which the substance is subjected to simple extension or in simple shear. Classifications on the basis of such experiments are presented in Table 1.2. As presented there, between elastic solids and Newtonian fluids, a succession of effects is shown having different forms which find application in drag reduction and drug delivery among other applications [10].

The constitutive equation for an incompressible and isotropic non-Newtonian fluid is given in Eq. (1.22). Fluids which obey this equation are called second-order Rivlin–Erickson fluids [42]:

$$\mathbf{T} = -p\mathbf{I} + \mu_1 \mathbf{A_1} + \mu_2 \left(\mathbf{A_1} \cdot \mathbf{A_1} \right) + \mu_3 \mathbf{A_2}, \tag{1.22}$$

where

$\mathbf{T}$ is the complete stress tensor,
$\mathbf{A_n}$ are Rivlin–Erickson tensors,
μ_1 is Newtonian viscosity,
μ_2 is cross-viscosity,
μ_3 is elastico-viscosity.

For flows characterized by only stress and rate of deformation, the most widespread form of constitutive equation is given in Eq. (1.23). This relation describes Stokesian fluids, and in certain instances Reiner–Rivlin fluids [10]:

$$\tau_{ij} = \left(-p + \mu' \mathbf{\nabla} \cdot \mathbf{V} \right) \delta_{ij} + \mu \delta_{ij} + \mu'' d_{ik} d_{kj}, \tag{1.23}$$

where μ'' is additional viscosity coefficient.

The constitutive equation for advanced Bingham fluids, or occasionally viscoplastic fluids, is [37]:

$$\tau_{ij} = \mathrm{T_{ij}} \quad \text{if } \mathrm{T_{ij}} \mathrm{T_{jk}} \leq \mathrm{T}^2, \text{ or}$$

$$\tau_{ij} = \mathrm{T_{ij}} + \left(-p + \mathbf{\nabla} \cdot \mathbf{V} \right) \delta_{ij} + \mu d_{ij} \quad \text{if } \frac{1}{2} \mathrm{T_{mn}} \mathrm{T_{mn}} > \mathrm{T}^2.$$

For the latter equation,

$$\mathrm{T_{ij}} = \frac{2T}{\sqrt{2\mathrm{T_{mn}}\mathrm{T_{mn}}}} d_{ij}, \tag{1.24}$$

where $\mathrm{T_{ij}}$ is yield stress tensor component, T is yield stress based on the von Mises yield criterion.

Table 1.2 Classification of non-Newtonian fluids [10].

Type of fluid	Classification	Stress/rate of deformation behavior	Examples
Elastic solids	Hookean	Linear stress–strain relation	Most solids below the yield stress
Plastic solids	Perfectly plastic	Strain continues with no additional stress	Ductile metals stressed above the yield point
	Bingham plastic	Behaves Newtonian when threshold in exceeded	Iron oxide suspensions
	Visco-plastic	Yield like the Bingham plastic, but the relation between stress and rate of deformation is not linear	Drilling mud, nuclear fuel slurries, mayonnaise, toothpaste, blood
	Yield dilatant[a]	Dilatant when threshold shear is exceeded	
	Visco-elastic	Exhibits both viscous and elastic effects	Egg white, polymer melts, and solutions
Power-law fluids	Shear thinning	Apparent viscosity reduces as shear rate increases	Some colloids, clay, milk, gelatin, blood, liquid cement, molten polystyrene, polyethylene oxide in water
	Dilatant or shear thickening	Apparent viscosity increases as shear rate increases	Concentrated solutions of sugar in water, suspensions of rice or cornstarch, solutions of certain surfactants
Time-dependent viscosity	Rheopectic	Apparent viscosity increases the longer stress is applied	Some lubricants
	Thixotropic	Apparent viscosity decreases the longer stress is applied	Nondrip paints, tomato ketchup
Electromagnetic	Electrorheologic	Becomes dilatant when an electric field is applied	Melted chocolate bars, single – or polycrystalline suspensions in insulating fluids
	Magnetorheo-logic	Becomes dilatant when an magnetic field is applied	Colloids with nanosize silica particles suspended in polyethylene glycol
Newtonian fluids		Linear stress-rate of deformation relationship	Water, air

a Dilatant here refers to shear thickening as stress increases.

1.3.4 The law of conservation of energy (the first law of thermodynamics)

From the first law of thermodynamics applied to the system, the sum of the work and heat supplied to the system will give rise to an increase in the energy of the system as presented mathematically by

$$dE_t = dQ + dW, \tag{1.25}$$

where

E_t is the total energy possessed by the system,
Q is heat added to the system, and
W is work done on the system.

For a flowing fluid particle, the total energy is the sum of the internal, kinetic, and potential energy. Therefore, the energy per unit volume in this case is

$$E_t = \rho \left(e + \frac{1}{2}V^2 - \boldsymbol{g} \cdot \boldsymbol{r} \right), \tag{1.26}$$

where

e is the internal energy per unit mass and
r is the displacement of the particle.

Writing the energy equation as a time rate of change, which follows the particle of a fluid, gives:

$$\frac{DE_t}{Dt} = \frac{DQ}{Dt} + \frac{DW}{Dt}, \tag{1.27}$$

or

$$\frac{DE_t}{Dt} = \rho \left(\frac{De}{Dt} + V\frac{DV}{Dt} - \boldsymbol{g} \cdot \boldsymbol{V} \right). \tag{1.28}$$

The heat and work transfer on a differential fluid element is shown in Fig. 1.5. For some materials, Fourier's law is the constitutive relation for thermal energy diffusion within the molecules of the material [8,10]. Assuming that heat transfer to the element is given by Fourier's law yields

$$\boldsymbol{q} = -k\nabla T. \tag{1.29}$$

Considering heat flow in the x-direction as shown in Fig. 1.5, the heat flow into the element is "$q_x dy dz$" and the heat flow out of the element is

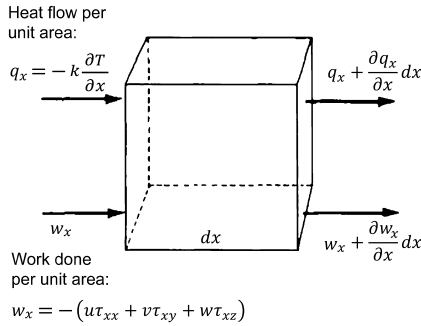$$\left(q_x + \frac{\partial q_x}{\partial x}dx \right) dy dz.$$

Figure 1.5 Exchange of heat and work on the left- and right-hand sides of an element of fluid.

Similarly, by obtaining the heat flow in the y- and z-directions, respectively, the net heat transfer to the element is

$$-\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) dx dy dz. \tag{1.30}$$

If internal heat generation is neglected, dividing Eq. (1.30) by the element of volume $dxdydz$ gives [43]

$$\frac{DQ}{Dt} = -\operatorname{div} q = +\operatorname{div}(k\nabla T). \tag{1.31}$$

From Fig. 1.5, the net rate of work done on the element of fluid per unit volume is [11]

$$\begin{aligned}
\frac{DW}{Dt} &= -\operatorname{div} w \\
&= \frac{\partial}{\partial x}\left(u\tau_{xx} + v\tau_{xy} + w\tau_{xz}\right) + \frac{\partial}{\partial y}\left(u\tau_{yx} + v\tau_{yy} + w\tau_{yz}\right) \\
&\quad + \frac{\partial}{\partial z}\left(u\tau_{zx} + v\tau_{zy} + w\tau_{zz}\right),
\end{aligned}$$

or

$$\frac{DW}{Dt} = \nabla \cdot \left(V \cdot \tau_{ij}\right). \tag{1.32}$$

Decomposing the expression above yields

$$\nabla \cdot \left(V \cdot \tau_{ij}\right) = V \cdot \left(\nabla \cdot \tau_{ij}\right) + \tau_{ij}\frac{\partial u_i}{\partial x_j}. \tag{1.33}$$

But from the momentum equation [8,43],

$$\rho\frac{DV}{Dt} = \rho g + \nabla \cdot \tau_{ij}, \tag{1.34}$$

so that the first term on the right-hand side of Eq. (1.5) can be written as [43]

$$\rho \left(V \frac{DV}{Dt} - \boldsymbol{g} \cdot \boldsymbol{V} \right).$$

Substituting the appropriate terms for $\frac{DE_t}{Dt}$ (Eq. (1.28)), $\frac{DQ}{Dt}$ (Eq. (1.31)), and $\frac{DW}{Dt}$ (Eq. (1.33)) into Eq. (1.27) gives a form of the first law of thermodynamics commonly used for fluid motion [43]:

$$\rho \frac{De}{Dt} = \text{div}\,(k\nabla T) + \tau_{ij} \frac{\partial u_i}{\partial x_j}. \tag{1.35}$$

A more well-known form of Eq. (1.35) can be obtained by noting that [8]

$$\tau_{ij} \frac{\partial u_i}{\partial x_j} = -p\,\text{div}\,V + \tau'_{ij} \frac{\partial u_i}{\partial x_j}, \tag{1.36}$$

where the elements of the viscous stress dyadic tensor for Newtonian fluids in the rectangular coordinate system are [8]

$$\tau'_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \delta_{ij}\,\text{div}\,V, \tag{1.37}$$

from the continuity equation (Eq. (1.2)), the following can be obtained:

$$p\,\text{div}\,V = -\frac{p}{\rho} \frac{D\rho}{Dt} = \rho \frac{D}{Dt} \left(\frac{p}{\rho} \right) - \frac{Dp}{Dt}. \tag{1.38}$$

Joining Eqs. (1.36) and (1.38) gives

$$\rho \frac{D}{Dt} \left(e + \frac{p}{\rho} \right) = \frac{Dp}{Dt} + \text{div}\,(k\nabla T) + \tau'_{ij} \frac{\partial u_i}{\partial x_j} = \frac{Dp}{Dt} + \text{div}\,(k\nabla T) + \mu\Phi. \tag{1.39}$$

The last term in Eq. (1.39) is the viscous dissipation term, and Φ is the dissipation function.

For a Newtonian fluid, the dissipation function in the Cartesian coordinate system is [3,8,38])

$$\Phi = 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right] + \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right]^2 + \left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right]^2$$
$$+ \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right]^2 - \frac{2}{3} \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right]^2. \tag{1.40}$$

The dissipation function in the cylindrical coordinate system is [3]

$$\Phi = 2 \left[\left(\frac{\partial v_r}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right)^2 + \left(\frac{\partial v_z}{\partial z} \right)^2 \right] + \left[r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right]^2$$

$$+ \left[\frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \right]^2 + \left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right]^2 - \frac{2}{3} \left[\frac{1}{r} \frac{\partial (r v_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \right]^2 .$$

$$(1.41)$$

In boundary-layer flows, the enthalpy is, in general, more useful than the internal energy; and the term $\frac{Dp}{Dt}$ in Eq. (1.39) is usually negligible, while the term p div V in Eq. (1.36) cannot be neglected. Normally, the viscous dissipation term is neglected unless the system has large velocity gradients [3].

1.3.5   *The second law of thermodynamics: entropy production*

The second law of thermodynamics is essentially [20] given by

$$\Delta S_{total} = \Delta S_{system} + \Delta S_{surrounding} \equiv S_{gen} > 0. \tag{1.42}$$

The entropy, S, gives an indication of the extent of molecular material randomness, and for a process to take place or for a device to function, S_{gen} (entropy production) must be greater than zero. Heat transfer causes the change of entropy. The greater the value of S_{gen}, the lower the efficiency of the device, process, or system [20]. The change of entropy is given by [21,43]

$$T ds = de + p dv = de - \frac{p}{\rho^2} d\rho. \tag{1.43}$$

For a particle of fluid, the rate of change of entropy is [21]

$$T \frac{DS}{Dt} = \frac{De}{Dt} - \frac{p}{\rho^2} \frac{D\rho}{Dt}. \tag{1.44}$$

Putting the internal energy and continuity equations into Eq. (1.44) gives [21]

$$\rho \frac{DS}{Dt} = -\frac{1}{T} \frac{\partial q_i}{\partial x_i} + \frac{\emptyset}{T} = -\frac{\partial}{\partial x_i} \left(\frac{q_i}{T} \right) - \frac{q_i}{T^2} \frac{\partial T}{\partial x_i} + \frac{\emptyset}{T}. \tag{1.45}$$

By utilizing Fourier's law of heat conduction, this equation becomes

$$\rho \frac{DS}{Dt} = -\frac{\partial}{\partial x_i} \left(\frac{q_i}{T} \right) + \frac{k}{T^2} \left(\frac{\partial T}{\partial x_i} \right)^2 + \frac{\emptyset}{T}. \tag{1.46}$$

The gain of entropy as a result of reversible heat transfer is the first term on the right-hand side of the equation. The second term represents the entropy production because of heat conduction while the last term represents the entropy production as a result of viscous heat generation. Since the second law of thermodynamics demands that the entropy production as a result of irreversible phenomena be positive, μ and k are greater than zero. For inviscid flow which does not conduct heat, entropy is conserved along the paths of the fluid particles [21].

1.4   Velocity slip and temperature jump

If the fluid that makes contact with a solid surface is a liquid, the molecules of the liquid are very closely packed and their mean free path is very small, so that the particles of a fluid which makes contact with the surface are basically in equilibrium with the surface. Thus, the fluid particles stick to the surface and get into thermal equilibrium with the surface. Both boundary conditions are referred to as no-slip and no-temperature jump conditions, respectively. For the case in which the fluid is a gas, and the molecules of the gas have large mean free path, the no-slip, no-temperature jump boundary conditions will not hold [36,43]. The slip velocity, u_w, is

$$u_w \approx \frac{3}{2} \frac{\mu}{\rho a} \frac{\tau_w}{\mu}, \tag{1.47}$$

where a is the speed of sound in the gas. Dividing by the free stream velocity, U, and rearranging gives

$$\frac{u_w}{U} \approx \frac{3}{4} \frac{U}{a} \frac{2\tau_w}{\rho U^2} = 0.75 M_a C_f, \tag{1.48}$$

where

M_a is Mach number of the free stream and
C_f is the flow's skin-friction coefficient.

For turbulent flow, C_f is not greater than 0.005, and this value decreases as the Mach number increases, so that it may be concluded that for turbulent boundary layer u_w is approximately equal to zero. Thus, the no-slip condition applies, and for laminar boundary layer, the skin friction coefficient is approximately [6]

$$C_f \approx 0.6 Re_x^{-\frac{1}{2}}, \tag{1.49}$$

where Re_x is the local Reynolds number. Combining Eqs. (1.48) and (1.49) gives for laminar boundary layer [31]

$$\frac{u_w}{U} \approx \frac{0.4 M a}{\sqrt{Re_x}}. \tag{1.50}$$

Thus, for large Mach numbers and small Reynolds numbers, considerable slip can be obtained. As the flow moves further downstream over the solid boundary, the value of Re_x increases, so that slip is no longer significant [2]. Similar to velocity slip, if the mean free path for the flow of a gas is large in comparison with the flow dimensions, the effect of temperature jump occurs. This effect occurs because some gas molecules on the solid surface do not come into thermal equilibrium with it [15,43]. The expression of the kinetic theory for the temperature jump, $T_{\text{gas}} - T_w$, is

$$T_{\text{gas}} - T_w \approx \left(\frac{2}{\alpha} - 1\right) \frac{2\gamma}{\gamma + 1} \frac{lk}{\mu c_p} \left(\frac{dT}{dy}\right)_w, \tag{1.51}$$

where

T_{gas} is the temperature of the gas,
T_w is the temperature of the solid surface,
l is the mean free path of gas molecules,
γ is the specific heat ratio, and
α is the thermal-accommodation coefficient, defined as follows [15]:

$$\alpha = \frac{E_i - E_r}{E_i - E_w}, \tag{1.52}$$

where

E_i is the energy of the molecules which strike the surface,
E_r is the energy of the molecules which were reflected from the surface, and
E_w is the energy which the molecules would possess if the molecules attained the temperature of the surface and had the same amount of energy as the surface.

The values of α must be obtained experimentally, and experimental results reveal that the value of α is approximately one [15,43]. Thus, taking the value of α as one and substituting appropriate values for l and $\frac{dT}{dy}$ (from Fourier's law); applying Reynolds analogy and taking $\gamma = 1.4$ for air, gives the following relation for the temperature jump divided by the driving temperature difference controlling the wall heat transfer, $T_r - T_w$:

$$\frac{T_{\mathrm{gas}} - T_w}{T_r - T_w} = 0.87 Ma C_f. \tag{1.53}$$

Therefore, it is seen that in turbulent flow, temperature jump is insignificant. In laminar flow, temperature jump is very small, except in regions that are very close to the leading edge of a flow which has a high Mach number. Usually, the no-slip and no-temperature jump boundary conditions are used in regular analysis of viscous flow of gases [25].

References

[1] B. Alkahtani, M.S. Abel, E.H. Aly, Analysis of fluid motion and heat transport on magneto-hydrodynamic boundary layer past a vertical power law stretching sheet with hydrody-namic and thermal slip effects, AIP Advances 5 (2015) 127228.1–127228.11.

[2] F.G. Awad, S.M.S. Ahamed, P. Sibanda, M. Khumalo, The effect of thermophoresis on unsteady Oldroyd-b nanofluid flow over stretching surface, PLoS ONE 10 (8) (2015) 1–23, https://doi.org/10.1371/journal.pone.0135914.

[3] R.B. Bird, W.E. Stewart, E.N. Lightfoot, Transport Phenomena, 2nd ed, Wiley, New York, 2002.

[4] Y.A. Çengel, J.M. Cimbala, Fluid Mechanics: Fundamentals and Applications, McGraw-Hill, Boston, 2006.

[5] J.G.M. Eggels, F. Unger, M.H. Weiss, J. Westerweel, R.J. Adrian, R. Friedrich, F.T.M. Nieuwstadt, Fully developed turbulent pipe flow: a comparison between direct numerical simulation and experiment, Journal of Fluid Mechanics 268 (1994) 175–209.

[6] R.W. Fox, A.T. McDonald, Introduction to Fluid Mechanics, 5th ed, Wiley, New York, 2001.

[7] A.Y. Gelfgat, Visualization of three-dimensional incompressible flows by quasi-two-dimensional divergence-free projections in arbitrary flow regions, Theoretical and Computational Fluid Dynamics 30 (4) (2016) 339–348.

[8] S.M. Ghiaasiaan, Convective Heat and Mass Transfer, Cambridge University Press, Cambridge, 2011.

[9] L.A. Glasgow, Transport Phenomena: An Introduction to Advanced Topics, Wiley, New Jersey, 2010.

[10] W.P. Graebel, Advanced Fluid Mechanics, Academic Press, Amsterdam, 2007.

[11] V.G. Gupta, J. Ajay, A.K. Jha, Convective effects on MHD flow and heat transfer between vertical plates moving in opposite direction and partially filled with a porous medium, Journal of Applied Mathematics and Physics 4 (2016) 341–358.

[12] N.G. Hadjiconstantinou, A.T. Patera, Heterogeneous atomistic-continuum representations for dense fluid systems, International Journal of Modern Physics 8 (1997) 967–976.

[13] R. Haldenwang, R. Kotze, R. Chhabra, Determining the viscous behavior of non-Newtonian fluids in a flume using a laminar sheet flow model and ultrasonic velocity profiling (UVP) system, Journal of the Brazilian Society of Mechanical Sciences and Engineering 34 (3) (2012) 276–284.

[14] S. Han, New conception in continuum theory of constitutive equation for anisotropic crystalline polymer liquids, Natural Science 2 (9) (2010) 948–958.

[15] J.P. Holman, Heat Transfer, 10th ed, McGraw-Hill, Boston, 2010.

[16] R.N. Ibragimov, Nonlinear viscous fluid patterns in a thin rotating spherical domain and applications, Physics of Fluids 23 (2011) 123102.1–123102.8.

[17] R.N. Ibragimov, D.E. Pelinovsky, Incompressible viscous fluid flows in a thin spherical shell, Journal of Mathematical Fluid Mechanics 11 (1) (2009) 60–90.

[18] D.D. Joseph, Potential flow of viscous fluids: historical notes, International Journal of Multiphase Flow 32 (2006) 285–310.

[19] W.M. Kays, M.E. Crawford, Convective Heat and Mass Transfer, 3rd ed, McGraw-Hill, New York, 1993.

[20] C. Kleinstreuer, Modern Fluid Dynamics: Basic Theory and Selected Applications in Macro- and Micro-Fluidics, Springer, Dordrecht, 2010.

[21] P.K. Kundu, I.M. Cohen, Fluid Mechanics, 2nd ed, Academic Press, San Diego, 2002.

[22] G.L. Leal, Advanced Transport Phenomena: Fluid Mechanics and Convective Transport Processes, Cambridge University Press, Cambridge, 2007.

[23] G.W. Milton, J.R. Willis, On modifications of Newton's second law and linear continuum elastodynamics, Proceedings of the Royal Society A (2007) 855–880.

[24] M. Miyan, Different phases of Reynolds equation, International Journal of Applied Research 2 (1) (2016) 140–148.

[25] B. Mohajer, V. Aliakbar, M. Shams, A. Moshfegh, Heat transfer analysis of a micro-spherical particle in the slip flow regime by considering variable properties, Heat Transfer Engineering 36 (6) (2015) 596–610.

[26] Y. Morinishi, O.V. Vasilyev, T. Ogi, Fully conservative finite difference scheme in cylindrical coordinates for incompressible flow simulations, Journal of Computational Physics 197 (2) (2004) 686–710.

[27] F.A. Morrison, An Introduction to Fluid Mechanics, Cambridge University Press, Cambridge, 2013.

[28] B.R. Munson, et al., Fundamentals of Fluid Mechanics, 7th ed, Wiley, New Jersey, 2013.

[29] Y. Nakayama, R.F. Boucher, Introduction to Fluid Mechanics, Butterworth–Heinemann, Oxford, 1999.

[30] D.A. Nelson, G.B. Jacobs, D.A. Kopriva, Effect of boundary representation on viscous, separated flows in a discontinuous-Galerkin Navier–Stokes solver, Theoretical and Computational Fluid Dynamics 30 (4) (2016) 363–385.

[31] A. Noor, R. Nazar, K. Jafar, I. Pop, Boundary-layer flow and heat transfer of nanofluids over a permeable moving surface in the presence of a coflowing fluid, Advances in Mechanical Engineering 6 (2014) 1–10.

[32] S.T. O'Connell, P.A. Thompson, Molecular dynamics-continuum hybrid computations: a tool for studying complex fluid flows, Physical Review E 52 (1995) 5792–5795.

[33] P.H. Oosthuizen, D. Naylor, An Introduction to Convective Heat Transfer Analysis, McGraw-Hill, New York, 1999.

[34] T.J. Pedley, Lectures on plankton and turbulence: introduction to fluid dynamics, Scientia Marina 61 (1) (1997) 7–24.

[35] P.J. Pritchard, Fox and McDonald's Introduction to Fluid Mechanics, 8th ed, Wiley, New Jersey, 2011.

[36] M.M. Rashidi, N.F. Mehr, Effects of velocity slip and temperature jump on the entropy generation in magnetohydrodynamic flow over a porous rotating disk, Journal of Mechanical Engineering 1 (3) (2012) 4–14.

[37] P. Saramito, A new constitutive equation for elastoviscoplastic fluid flows, Journal of Non-Newtonian Fluid Mechanics 145 (2007) 1–14.

[38] H. Schlichting, Boundary-Layer Theory, 7th ed, McGraw-Hill, New York, 1979.

[39] J.H. Spurk, N. Aksel, Fluid Mechanics, 2nd ed, Springer-Verlag, Berlin, 2008.

[40] V.L. Streeter, Fluid Mechanics, 3rd ed, McGraw-Hill, New York, 1962.

[41] R. Verzicco, P. Orlandi, A finite-difference scheme for three-dimensional incompressible flows in cylindrical coordinates, Journal of Computational Physics 123 (1996) 402–414.

[42] Z.U. Warsi, Fluid Dynamics: Theoretical and Computational Approaches, 3rd ed, CRC Press, Boca Raton, 2006.

[43] F.M. White, Viscous Fluid Flow, 2nd ed, McGraw-Hill, New York, 1991.

[44] P.K. Yeung, Lagrangian investigations of turbulence, Annual Review of Fluid Mechanics 34 (2002) 115–142.