A STUDY OF SOME COMPUTATIONAL ALGORITHMS FOR SOLVING FIRST ORDER INITIAL VALUE PROBLEMS

BY

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CERTIFICATION

This is to certify that this work was carried out by IYASELE KELVIN EHIZOJIE with Matric No MTH/11/0319 under my supervision

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H.O.D

DEDICATION

This work being the first significant product of my academic pursuit is first and foremost dedicated to Almighty God, the I AM THAT I AM who has made all things possible. I also dedicate this work to the best parents in the world, Mr. and Mrs. Charles Iyasele for all the endless love, care and support I've always enjoyed from them and I pray that God will give them longlife to reap the fruit of their labour. Also, to my siblings and these wonderful set of people God has blessed me with Rev Fr. Felix-Kingsley Obialo, Rev Fr. Alfred Omoleye, Rev Fr. Edward Oladele and Mr. and Mrs. S.A. Adesina for thier prayers and love.

Finally, to my late bossom friend Afunleyin Abayomi Omoba(1990-2015) I will always miss you.

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Abstract

This work takes a look at different computational algorithms used in solving initial value problems and how these algorithms are derived from Taylor's series. It also made use of the Euler and Runge-Kutta method to solve initial value problems in order to compare the performance of the two methods.

CHAPTER ONE

1. INTRODUCTION

DEFINITION OF TERMS

1.1 ALGORITHM

An algorithm is an effective method that can be expressed within a finite amount of space and in a well-defined formal language for calculating a function. Starting from an initial state and initial input (sometimes empty i.e 0), the instructions describe a computation that, when executed, proceeds through a finite number of well-defined successive states, eventually producing an "output" and terminating at a final ending state.
It is a step by step procedure used to solve a mathematical computation.

1.2 ONE-STEP METHOD

One-Step algorithms are characterised by the fact that they have no"memory". I use the word "memory" because one-step method algorithm treats each new time step computation as an initial value problem and does not use any previously computed solution points. e.g to compute the solution point Y_{j+1}, a one step algorithm uses the solution point Y_j as initial value, and does not use any previously computed solution points. An example of the one-step method is the Euler method.

1.3 RUNGE-KUTTA METHOD

The Runge-Kutta methods are a large class of one-step algorithms.
Let b_i, a_{ij} $(i,j,=1,\ldots,\ s\ )$ be real numbers and let $c_i = \sum_{j=1}^{s}$. An s-stage Runge-Kutta method is given by

$$k_i = f(t_0 + c_i h, y_0 + h\sum_{j=1}^{s} a_{ij}k_j), i = 1,\ldots,s \tag{1}$$

$y_i = y_0 +$ h $\sum_{j=1}^{s} b_i k_i$.

1.4 EULER METHOD

The simplest solution for an ordinary differential equation with an initial value problem is Euler's method, given by

$$Y_0 = \tau$$

$$Y_{j+1} = Y_j + h_j f(t_j, Y_j), j = 0, 1, \ldots, p-1. \quad (2)$$

where $h_j = t_{j+1} - t_j$ is the time step.
Euler's method has the basic features common to all solution algorithms. The algorithm starts with the given initial value $Y_0 = y(t_0) = \tau$, and then marches forward in time, computing the sequence of approximate solution values $Y_0 = y(t_0)$, $Y_1 \approx y(t_1)$, $Y_2 \approx y(t_2)$, $\ldots$, $Y_p \approx y(t_p)$ in order.

1.5 ERROR

This is the difference between the exact mathematical solution and the approximate solution obtained when simplifications are made to the mathematical equations to make them more amenable to calculation.

1.6 TRUNCATION ERROR

In computation, truncation error is the discrepancy that arises from executing a finite number of steps to approximate an infinite process. For example,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + x^2/2! + x^3/3! + \ldots$$

$$e^x = 1 + x + x^2/2!$$

$$k_1 = e^x + 1 + x + x^2/2 + x^3/3! + \ldots$$

$$k_2 = 1 + x + x^2/2!$$

Trucation error $= k_1 - k_2 = 1 + x + x^2/2! + x^3/31 + \ldots (1 + x + x^2/2!) = x^3 + x^4/4! \ldots$

$$T.E = \sum_{\infty}^{n=3} \frac{x^n}{n!}$$

Note: Truncation error could be either local or global. It is local if the truncation error τ_n is the error that our increment function, A , causes during a single iteration, assuming perfect knowledge of the true solution at the previous iteration. More formally, the local truncation error, τ_n , at step n is computed from the difference between the left- and the right-hand side of the equation for the increment $y_n \approx y_{n-1} + hA(t_{n-1}, y_{n-1}, h, f)$: $\tau_n = y(t_n) - y(t_{n-1}) - hA(t_{n-1}, y(t_{n-1}), h, f)$.

The numerical method is consistent if the local truncation error is $o(h)$ (this means that for every $\varepsilon > 0$ there exists an H such that $|\tau_n| < \varepsilon h$ for all $h < H$). If the increment function A is differentiable, then the method is consistent if, and only if,$A(t, y, 0, f) = f(t, y)$.

Furthermore, we say that the numerical method has order p if for any sufficiently smooth solution of the initial value problem, the local truncation error is $O(h^{p+1})$ (meaning that there exist constants C and H such that $|\tau_n| < Ch^{p+1}$ for all $h < H$).

The truncation is global if the truncation error is the accumulation of the local truncation error over all of the iterations, assuming perfect knowledge of the true solution at the initial time step. More formally, the global truncation error,e_n , at time t_n is defined by:

$$e_n = y(t_n) - y_n = y(t_n) - \Big(y_0 + hA(t_0, y_0, h, f) + hA(t_1, y_1, h, f) + \\ \cdots + hA(t_{n-1}, y_{n-1}, h, f)\Big).$$

The numerical method is convergent if global truncation error goes to zero as the step size goes to zero; in other words, the numerical solution converges to the exact solution: $\lim_{h\to 0} \max_n |e_n| = 0$.

CHAPTER TWO

2. REVIEW OF SOME EXISTING METHODS

EULER'S METHODS

Leonhard Euler was one of the giants of 18th Century mathematics. Like the Bernoullis, he was born in Basel, Switzerland, and he studied for a while under Johann Bernoulli at Basel University. But, partly due to the overwhelming dominance of the Bernoulli family in Swiss mathematics, and the difficulty of finding a good position and recognition in his hometown, he spent most of his academic life in Russia and Germany, especially in the burgeoning St. Petersburg of Peter the Great and Catherine the Great.

Despite a long life and thirteen children, Euler had more than his fair share of tragedies and deaths, and even his blindness later in life did not slow his prodigious output - his collected works comprise nearly 900 books and, in the year 1775, he is said to have produced on average one mathematical paper every week - as he compensated for it with his mental calculation skills and photographic memory (for example, he could repeat the Aeneid of Virgil from beginning to end without hesitation, and for every page in the edition he could indicate which line was the first and which the last).[1]

Today, Euler is considered one of the greatest mathematicians of all time. His interests covered almost all aspects of mathematics, from geometry to calculus to trigonometry to algebra to number theory, as well as optics, astronomy, cartography, mechanics, weights and measures and even the theory of music.

Much of the notation used by mathematicians today - including e,i, $f(x)$, $\sum$, and the use of a, b and c as constants and x, y and z as unknowns - was either created, popularized or standardized by Euler. His efforts to standardize these and other symbols (including and the trigonometric functions) helped to internationalize mathematics and to encourage collaboration on problems.

[1]The Aeneid is a Latin epic poem, written by Virgil between 29 and 19 BC, that tells the legendary story of Aeneas, a Trojan who travelled to Italy, where he became the ancestor of the Romans. It comprises 9,896 lines in dactylic hexameter.

He even managed to combine several of these together in an amazing feat of mathematical alchemy to produce one of the most beautiful of all mathematical equations, $e^{i\pi} = -1$, sometimes known as Eulers Identity. This equation combines arithmetic, calculus, trigonometry and complex analysis into what has been called "the most remarkable formula in mathematics", "uncanny and sublime" and "filled with cosmic beauty", among other descriptions. Another such discovery, often known simply as Eulers Formula, is $e^{ix} = cosx + isinx$. In fact, in a recent poll of mathematicians, three of the top five most beautiful formulae of all time were Eulers. He seemed to have an instinctive ability to demonstrate the deep relationships between trigonometry, exponentials and complex numbers.

The discovery that initially sealed Eulers reputation was announced in 1735 and concerned the calculation of infinite sums. It was called the Basel problem after the Bernoullis had tried and failed to solve it, and asked what was the precise sum of the of the reciprocals of the squares of all the natural numbers to infinity i.e. $1/1^2 + 1/2^2 + 1/3^2 + 1/4^2 \ldots$ (a zeta function using a zeta constant of 2). Eulers friend Daniel Bernoulli had estimated the sum to be about $1\frac{3}{5}$, but Eulers superior method yielded the exact but rather unexpected result of $\frac{\pi^2}{6}$. He also showed that the infinite series was equivalent to an infinite product of prime numbers, an identity which would later inspire Riemanns investigation of complex zeta functions.

Also in 1735, Euler solved an intransigent mathematical and logical problem, known as the Seven Bridges of Knigsberg Problem, which had perplexed scholars for many years, and in doing so laid the foundations of graph theory and presaged the important mathematical idea of topology. The city of Knigsberg in Prussia (modern-day Kaliningrad in Russia) was set on both sides of the Pregel River, and included two large islands which were connected to each other and the mainland by seven bridges. The problem was to find a route through the city that would cross each bridge once and only once.

In fact, Euler proved that the problem has no solution, but in doing so he made the important conceptual leap of pointing out that the choice of route within each landmass is irrelevant and the only important feature is the sequence of bridges crossed. This allowed him to reformulate the problem in abstract terms, replacing each land mass with an abstract node and each bridge with an abstract connection. This resulted in a mathematical structure called a graph, a pictorial representation made up of points (vertices) connected by non-intersecting curves (arcs), which may be distorted in any way without changing the graph itself. In this way, Euler was able to deduce that, because the four land masses in the original problem are touched by an odd number of bridges, the existence of a walk traversing each bridge once only inevitably leads to a contradiction. If Knigsberg had had one fewer bridges, on the other hand, with an even number of bridges leading to each piece of land, then a solution would have been possible.

The list of theorems and methods pioneered by Euler is immense, and largely outside the scope of an entry-level study such as this, but mention could be made of just some of them: the demonstration of geometrical properties such as Eulers Line and Eulers Circle; the definition of the Euler Characteristic (chi) for the surfaces of polyhedra, whereby the number of vertices minus the number of edges plus the number of faces always equals 2 (see table at right); a new method for solving quartic equations; the Prime Number Theorem, which describes the asymptotic distribution of the prime numbers; proofs (and in some cases disproofs) of some of Fermats theorems and conjectures; the discovery of over 60 amicable numbers (pairs of numbers for which the sum of the divisors of one number equals the other number), although some were actually incorrect; a method of calculating integrals with complex limits (foreshadowing the development of modern complex analysis); the calculus of variations, including its best-known result, the Euler-Lagrange equation; a proof of the infinitude of primes, using the divergence of the harmonic series; the integration of Leibniz's differential calculus with Newton's Method of Fluxions into a form of calculus we would recognize today, as well as the development of tools to make it easier to apply calculus to real physical problems etc.

In 1766, Euler accepted an invitation from Catherine the Great to return to the St. Petersburg Academy, and spent the rest of his life in Russia. However, his second stay in the country was marred by tragedy, including a fire in 1771 which cost him his home (and almost his life), and the loss in 1773 of his dear wife of 40 years, Katharina. He later married Katharina's half-sister, Salome Abigail, and this marriage would last until his death from a brain hemorrhage in 1783.

The main objective of Euler's method in solving initial value prolem is to obtain approximations to the well-posed initial value problem

$$\frac{dy}{dt} = f(t, y), a \leq t \leq b, y(a) = \alpha \tag{3}$$

A continuous approximation to the solution will not be obtained; instead approximations to y will be generated at various values, called mesh points, in the interval [a,b]. Once the approximate solution is obtained at the points, the approximate solution at other points in the interval can be found by interpolation.

To do this, we first make the stipulation that the mesh points are equally distributed throughout the interval [a,b]. This condition is ensured by choosing a positive integer N and selecting the mesh points

$$t_i = a + ih, \text{ for each } i = 0, 1, 2, \ldots, N.$$

The common distance between the points $h = \frac{(b-a)}{N} = t_{i+1} - t_i$ is called the step size.

We now use Taylor's theorem to derive the Euler's method.

Suppose that $y(t)$, the unique solution to (3), $y(a) = \alpha$ has two continuous derivatives on [a,b], so that for each $i = 0, 1, 2, \ldots, N-1$,

$$y(t_{i+1}) = y(t_i) + y(t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2}y''(\xi_i) \tag{4}$$

for some number ξ_i in (t_i, t_{i+1}). Therefore, because $h = t_{i+1} - t_i$, we have

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(\xi) \tag{5}$$

and, because $y(t)$ satisfies the differential eqquation $y' = f(t, y)$, we write

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}y''(\xi_i)$$

Euler method constructs $w_0 \approx y(t_i)$ for each $i = 1, 2, \ldots, N$, by deleting the remainder term. Thus Euler's method is

$$w_0 = \alpha$$

$\mathrm{w}_{i+1} = w_i + hf(t_i, w_i)$, (6)for each $i = 0, 1, \ldots, N-1$

This equation is called the difference equation associated with Euler's method.

RUNGE-KUTTA METHODS

Runge-Kutta methods compute approximations Y_i to $y_i = y(x_i)$, with initial values $Y_0 = y_0 = \alpha$, where $x_i = a + ih, i \in Z^+$. Again, using Taylor series expansion

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2 y''_n + \ldots + \frac{1}{p!}h^p y_n^{(p)} + 0(h^p + 1) \tag{7}$$

so if we term $f(x_n, y_n) = f_n$ etc. :

$$y_{n+1} = y_n + hf_n + \frac{1}{2}h^2(\frac{df}{dx})_n + \ldots + \frac{1}{p!}h^p(\frac{d^{p-1}f}{dx^{p-1}})_n + 0(h^{p+1}) \tag{8}$$

h is a non-negative real constant called the *step length* of the method. To obtain a q-stage Runge-Kutta method (q function evaluations per step) we let

$$Y_{n+1} = Y_n + h\phi(x_n, Y_n; h), \tag{9}$$

where

$$\phi(x_n, Y_n; h) = \sum_{i=1}^{q} w_i k_i, \tag{10}$$

so that

$$Y_{n+1} = Y_n + h\sum_{i=1}^{q} w_i k_i, \tag{11}$$

with

$$k_i = f(x_n + h\alpha_i, Y_n + h\sum_{j=1}^{i-1} \beta_{ij} k_j) \tag{12}$$

and $\alpha_i = 0$ for an explicit method or

$$k_i = f(x_n + h\alpha_i, Y_n + h\sum_{j=1}^{q} \beta_{ij} k_j) \tag{13}$$

for an implicit method. for an explicit method, Eq.(8) can be solved for each k_i in turn, but for an implicit method, Eq.(9) requires the solution of a nonlinear system of k_is at each step. The set of explicit method may be regarded as a subset of the set of implicit methods with $\beta_{ij} = 0, j \geq i$. Explicit methods are obviously more efficient to use, but we shall see that implicit methods do have advantages in certain circumstances.

For convenience, the coefficients α, β, and ω of the Runge-Kutta method can be written in the form of a *Butcher array*:

$$\begin{array}{c|ccccc} 0 & & & & & \\ c_2 & a_{21} & & & & \\ \vdots & \vdots & \ddots & & & \\ c_s & a_{s1} & a_{s2} & \cdots & a_{s,s-1} & \\ \hline & b_1 & b_2 & \cdots & b_{s-1} & b_s \end{array} = \begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

Table1– Butcher Tableau

HEUN'S METHODS

In mathematics and computational science, Heun's method may refer to the improved or modified Euler's method (that is, the explicit trapezoidal rule), or a similar two-stage RungeKutta method. It is named after Karl Heun and is a numerical procedure for solving ordinary differential equations (ODEs) with a given initial value. Both variants can be seen as extensions of the Euler method into two-stage second-order RungeKutta methods.

The procedure for calculating the numerical solution to the initial value problem via the improved Euler's method is:

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0, \tag{14}$$

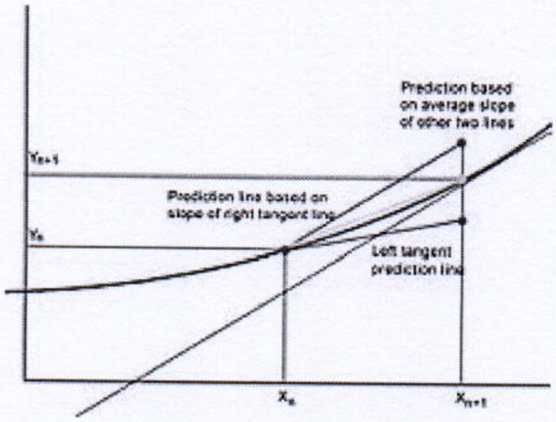
by way of Heun's method, is to first calculate the intermediate value $\tilde{y}_{i+1}$ and then the final approximation y_{i+1} at the next integration point.

$$\begin{aligned} \tilde{y}_{i+1} &= y_i + hf(t_i, y_i) \\ y_{i+1} &= y_i + \frac{h}{2}[f(t_i, y_i) + f(t_{i+1}, \tilde{y}_{i+1})], \end{aligned} \tag{15}$$

where h is the step size and $t_{i+1} = t_i + h$.

Eulers method is used as the foundation for Heuns method. Euler's method uses the line tangent to the function at the beginning of the interval as an estimate of the slope of the function over the interval, assuming that if the step size is small, the error will be small. However, even when extremely small step sizes are used, over a large number of steps the error starts to accumulate and the estimate diverges from the actual functional value.

Where the solution curve is concave up, its tangent line will underestimate the vertical coordinate of the next point and vice versa for a concave down solution. The ideal prediction line would hit the curve at its next predicted point. In reality, there is no way to know whether the solution is concave-up or concave-down, and hence if the next predicted point will overestimate or underestimate its vertical value. The concavity of the curve cannot be guaranteed to remain consistent either and the prediction may overestimate and underestimate at different points in the domain of the solution. Heuns Method addresses this problem by considering the interval spanned by the tangent line segment as a whole. Taking a concave-up example, the left tangent prediction line underestimates the slope of the curve for the entire width of the interval from the current point to the next predicted point. If the tangent line at the right end point is considered (which can be estimated using Eulers Method), it has the opposite problem. The points along the tangent line of the left end point have vertical coordinates which all underestimate those that lie on the solution curve, including the right end point of the interval under consideration. The solution is to make the slope greater by some amount. Heuns Method considers the tangent lines to the solution curve at both ends of the interval, one which overestimates, and one which underestimates the ideal vertical coordinates. A prediction line must be constructed based on the right end point tangents slope alone, approximated using Euler's Method. If this slope is passed through the left end point of the interval, the result is evidently too steep to be used as an ideal prediction line and overestimates the ideal point. Therefore, the ideal point lies approximately half way between the erroneous overestimation and underestimation, the average of the two slopes. Heun's Method.



A diagram depicting the use of Heun's method to find a less erroneous prediction when compared to the lower order Euler's Method

Eulers Method is used to roughly estimate the coordinates of the next point in the solution, and with this knowledge, the original estimate is re-predicted or corrected. Assuming that the quantity $f(x, y)$ on the right hand side of the equation can be thought of as the slope of the solution sought at any point (x, y) , this can be combined with the Euler estimate of the next point to give the slope of the tangent line at the right end-point. Next the average of both slopes is used to find the corrected coordinates of the right end interval. Derivation

$$Slope_{left} = f(x_i, y_i)$$
$$Slope_{right} = f(x_i + h, y_i + hf(x_i, y_i))$$
$$Slope_{ideal} = \frac{1}{2}(Slope_{left} + Slope_{right})$$

Using the principle that the slope of a line equates to the rise/run, the coordinates at the end of the interval can be found using the following formula:

$$Slope_{ideal} = (\Delta y/h)$$
$$\Delta y = h(Slope_{ideal})$$

$$x_{i+1} = x_i + h, y_{i+1} = y_i + \Delta y$$

$$y_{i+1} = y_i + hSlope_{ideal}$$

$$y_{i+1} = y_i + \frac{1}{2}h(Slope_{left} + Slope_{right})$$

$$y_{i+1} = y_i + \frac{h}{2}(f(x_i, y_i) + f(x_i + h, y_i + hf(x_i, y_i))) \tag{16}$$

The accuracy of the Euler method improves only linearly with the step size is decreased, whereas the Heun Method improves accuracy quadratically. The scheme can be compared with the implicit trapezoidal method, but with $f(t_{i+1}, y_{i+1})$ replaced by $f(t_{i+1}, \tilde{y}_{i+1})$ in order to make it explicit.$\tilde{y}_{i+1}$ is the result of one step of Euler's method on the same initial value problem. So, Heun's method is a predictor-corrector method with forward Euler's method as predictor and trapezoidal method as corrector.

CHAPTER THREE

3. IMPLEMENTATION OF RUNGE-KUTTA AND EULER METHOD

We now proceed to use the Runge-Kutta and the Euler methods to solve an initial value problem.

Example 1: Use the 4th order Runge-Kutta method with $h = 0.1$ to find the approximate solution for $y(1.1)$, for the initial value problem

$$\frac{dy}{dx} = 2xy, y(1) = 1$$

The Runge-Kutta method is given by

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + (\frac{1}{6})(k_1 + 2k_2 + 2k_3 + k_4) \quad (17)$$

where

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$$

$$k_3 = hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

We have dy/dx = f(x,y) = 2xy.

If you require the 4th order approximation the formula will be:

$$y(x_0 + h) = y(x_0) + (1/6)[k_1 + 2k_2 + 2k_3 + k_4]$$

where:

$$k_1 = h * f(x_0.y_0) = 0.1(2)(1)(1) = 0.2$$

$$k_2 = h * f(x_0 + h/2, y_0 + k_1/2) = 0.1(2)(1.05)(1.1) = 0.231$$

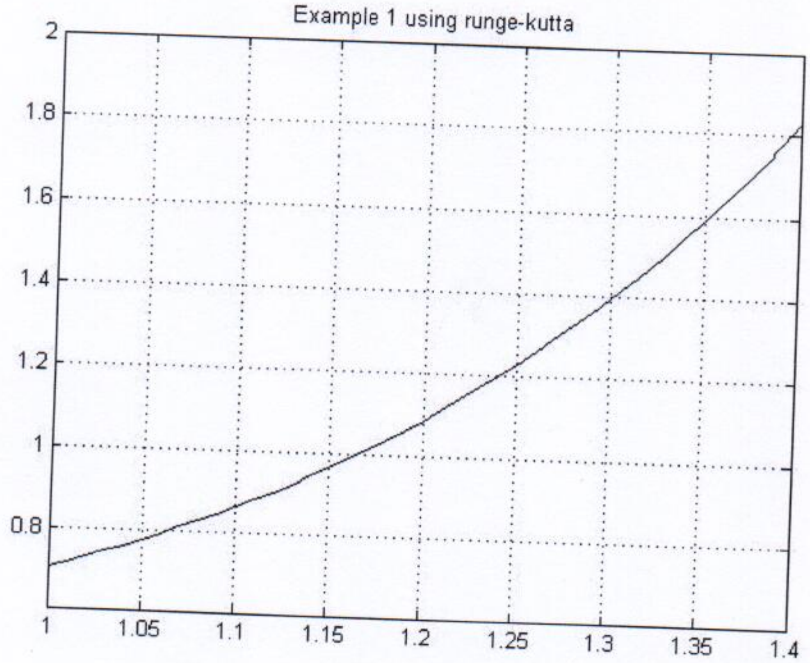
$$k_3 = h * f(x_0 + h/2, y)0 + k_2/2) = 0.1(2)(1.05)(1.1155) = 0.234255$$
$$k_4 = h * f(x_0 + h, y)0 + k_3) = 0.1(2)(1.1)(1.234255) = 0.2715361$$

and so:

$$y(1.1) = 1 + (1/6)[0.2 + 2(0.231) + 2(0.234255) + 0.2715361]$$
$$= 1.23367435$$

MATLAB CODE FOR SOLVING RUNGE-KUTTA METHOD

```
function z = myfun3(t,y)
z= 2*y*t;
end
[tv1 z1] = ode45 ('myfun3', [1,1.4],0.7);
plot (tv1, z1)
grid
```



graph1 showing result to example1 using rungekutta method

We can compare this with the exact solution to the problem.

$$\int dy/y = \int 2xdx$$

and integrating:

$$ln(y) = x^2 + C$$

and $y = 1$ when $x = 1$

$$0 = 1 + C$$

and so

$$C = -1$$

. Therefore:

$$ln(y) = x^2 - 1$$

$$y = e^{(x^2-1)}$$

When $x = 1.1$,this gives:

$$y = e^{(1.1^2-1)}$$

$$= e^{0.21}$$

$$= 1.23367806$$

The error term is given by;

$$\text{Error= Actual value} - \text{Approximate value}$$

Hence,

$$Error = 1.23367806 - 1.23367435$$
$$= 0.00000371$$

We then procced to use Euler method to solve the same initial value problem $\frac{dy}{dx} = 2xy, y(1) = 1$ with $h = 0.1$ to find the approximate solution for $y(1.1)$.

The Euler method is given by

$$w_o = \alpha$$
$$w_{i+1} = w_i + hf(t_i, w_i),$$

for each $i = 0, 1, \ldots, N - 1$.

Given that $\frac{dy}{dx} = 2xy \quad y(1) = 1 \quad h = 0.1 \quad N = 4 \quad x_0 = 1 \ and \ y_0 = 1$

$$x_{n=1} = x_n + h$$
$$y_{n+1} = y_n + hf(x, y)$$

When $n = 0$, we have that

$$x_1 = x_{0+1} = x_1 + h = 1 + 0.1 = 1.10000$$
$$y_1 = y_{o+1} = y_o + hf(x_o, y_o)$$
$$= 1 + 0.1(1)(1)$$
$$= 1.1000$$

Hence, $x_1 = 1.1000 \ and \ \mathrm{y}_1 = 1.1000$

When $n = 1$, we have that

$$x_2 = x_{1+1} = x_1 + h = 1.1000 + 0.1 = 1.2000$$
$$y_2 = y_{1+1} = y_1 + hf(x_1, y_1)$$
$$= 1.1000 + 0.1(1.1000)(1.1000)$$
$$= 1.2210$$

Hence, $x_2 = 1.2000$ and $y_2 = 1.2210$

When $n = 2$, we have that

$$x_3 = x_{2+1} = x_2 + h = 1.2000 + 0.1 = 1.3000$$

$$y_3 = y_{2+1} = y_2 + hf(x_2, y_2)$$
$$= 1.2210 + 0.1(1.2000)(1.2210)$$
$$= 1.36752$$

Hence, $x_3 = 1.3000$ and $y_3 = 1.36752$

Again, when $n = 3$, we have that

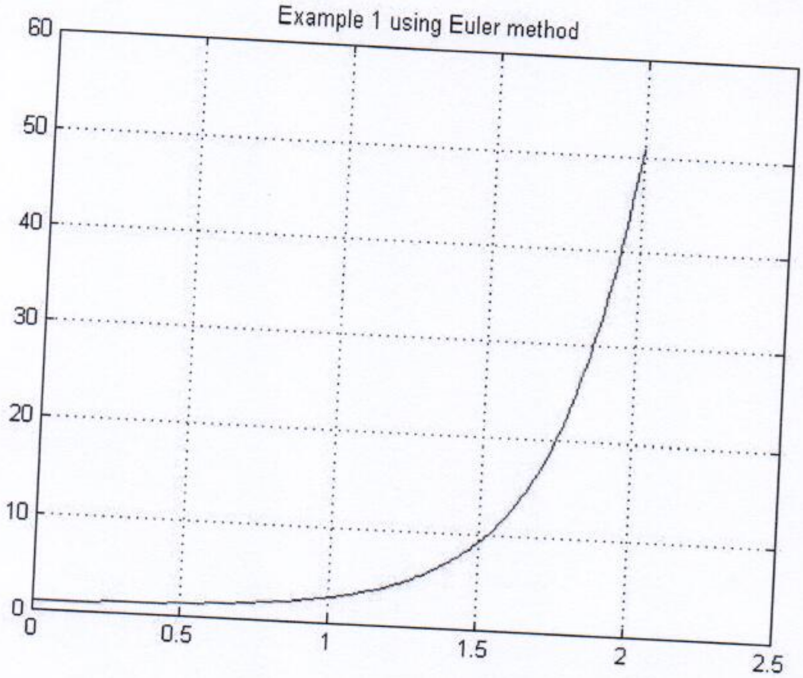
$$x_4 = x_{3+1} = x_3 + h + 1.3 + 0.1 = 1.4000$$

$$y_4 = y_{3+1} = y_3 + hf(x_3, x_3)$$
$$= 1.36752 + 0.1(1.3)(1.36752)$$
$$= 1.5452976$$

Hence, $x_4 = 1.4000$ and $y_4 = 1.5452976 \approx 1.5452$

MATLAB CODES FOR EULER METHOD

```
t=zeros(201,1);
y=zeros(201,1);
t(1)=0;
y(1)=1;
for i=1:200
t(i+1)=t(i)+0.01;
y(i+1)=y(i)+0.01*(y(i)*2*t(i));
end
plot(t,y)
grid
title (Example 1 using Euler method )
```



graph2 showing result of euler method

The table for the approximate value of y(x) is shown below

N	x_n	$approximate$	y_n	$approximate$
0	x_1	1.1000	y_1	1.1000
1	x_2	1.2000	y_2	1.2210
2	x_3	1.3000	y_3	1.36752
3	x_4	1.4000	y_4	1.5452976

Table 2 showing the points and their approximate values

The error again, is given by ;

Error= Actual value − Approximate value

Hence from our previous workings, we have our actual value to be 1.23367806

.

Therefore,

$$Error = 1.23367806 - 1.5452976$$
$$= -0.31161954 \approx -0.3116$$

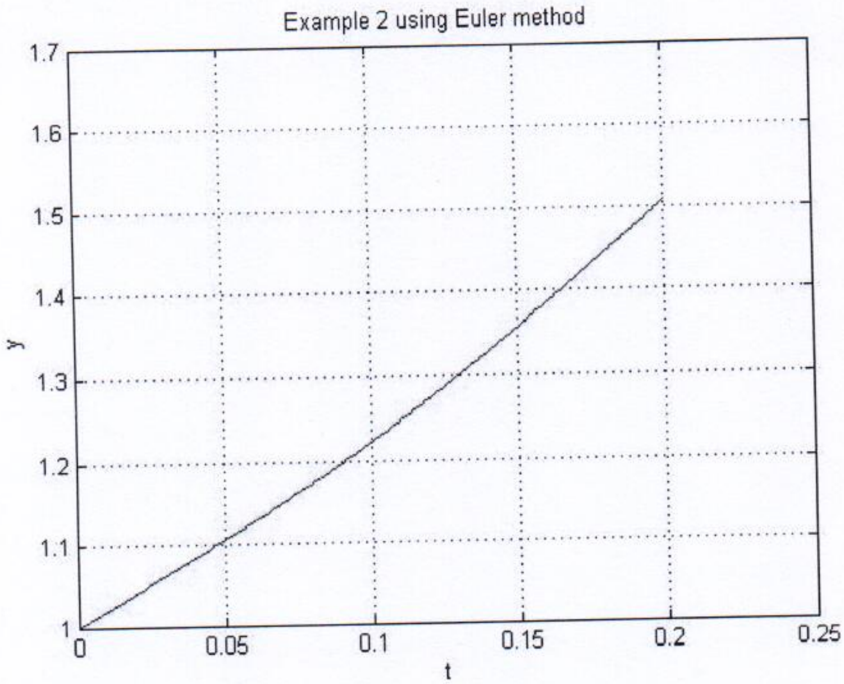
Example 2: We solve again for

$$y' = 1 + y^2, \quad y(0) = 1$$

Do this on the command window

MATLAB CODE FOR EULER METHOD

```
t=zeros(201,1);
y=zeros(201,1);
t(1)=0;
y(1)=1;
for i=1:200
t(i+1)=t(i)+0.001;
y(i+1)=y(i)+0.001*((y(i).^2) + 1);
end
plot(t,y)
```

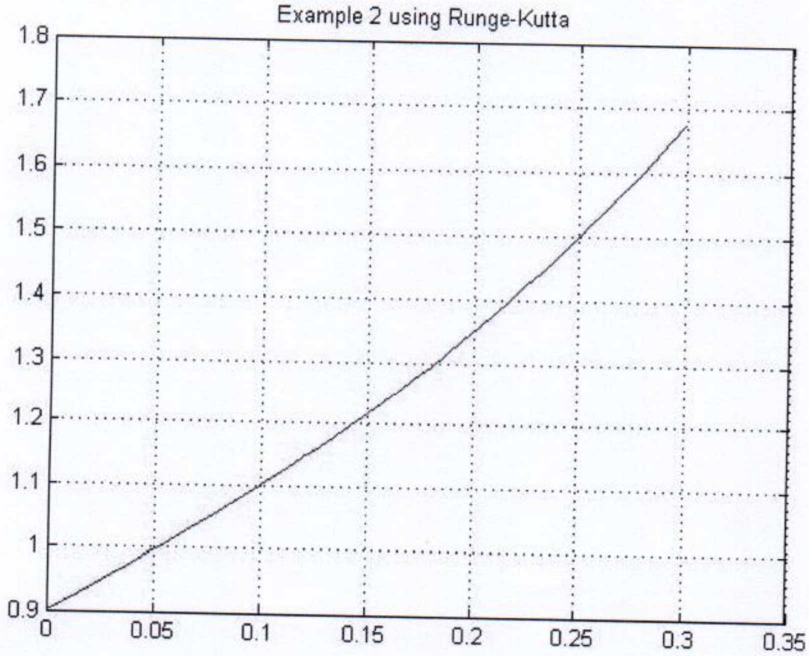


graph 3 showing the result using euler method

We then solve the same problem using Runge-kutta method

MATLAB CODES FOR RUNGE-KUTTA METHOD

```
function z = myfun1(t,y)
[tv1 z1] = ode45('myfun1', [0,0.30], 0.9);
plot(tv1 ,z1 ,'o');
title ('Example 2 using Runge-Kutta');
grid
function z = myfun1(t,y)
end
z = 1 + y.^2;
end
```



graph 4 showing result to example 2 using runge-kutta method

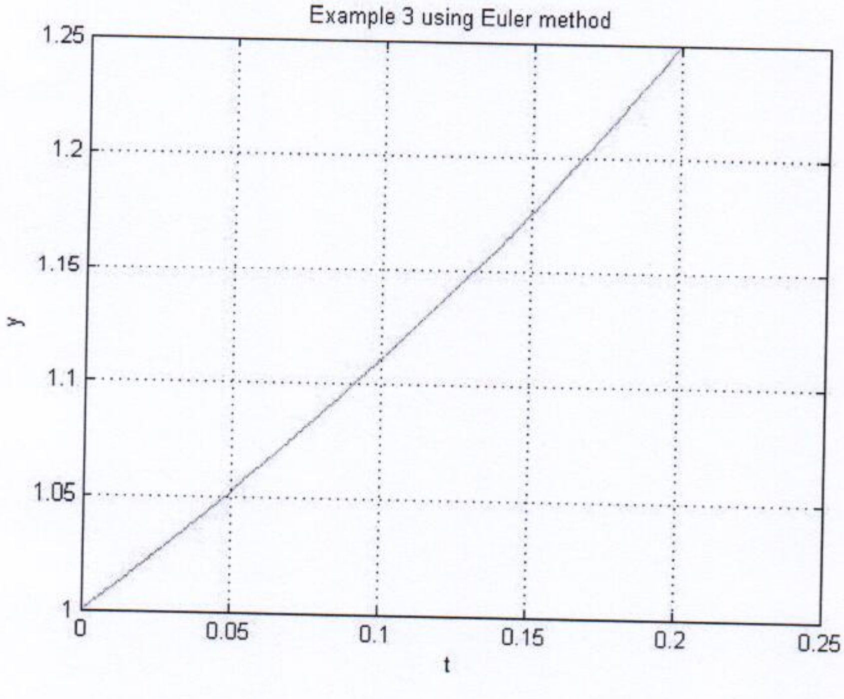
Example 3: Again solve for another problem

$$y' = y^2 \quad y(0) = 1$$

Do this on command window

MATLAB CODE FOR EULER METHOD

```
t=zeros(201,1);
y=zeros(201,1);
t(1)=0;
y(1)=1;
for i=1:200
t(i+1)=t(i)+0.001;
y(i+1)=y(i)+0.001*(y(i).^2);
end
plot(t,y,'g -')
grid
title (Example 3 using Euler method)
```

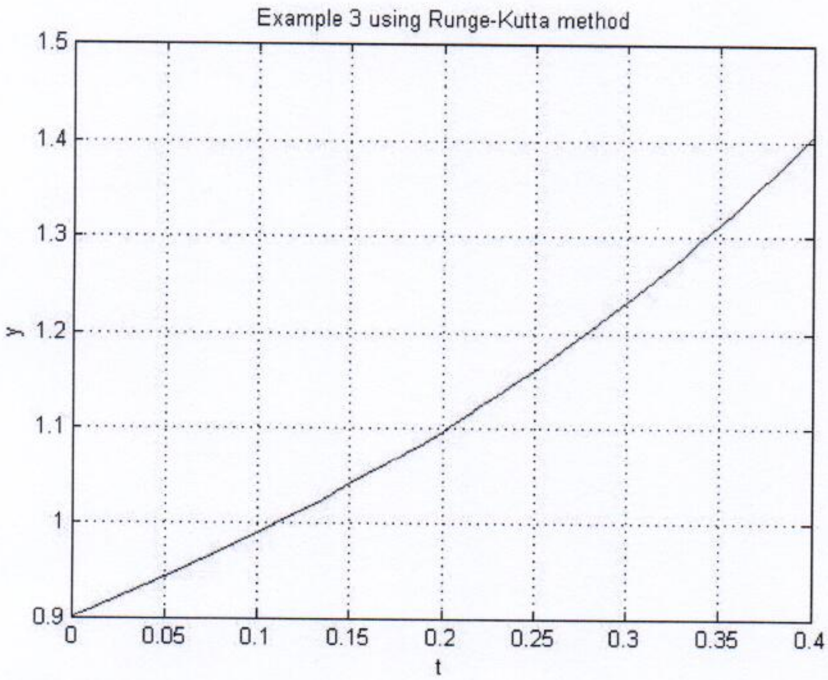


graph 5 showing result to example 3 using euler method

Solution to Example 3 using Runge-kutta method

MATLAB CODES FOR RUNGE-KUTTA METHOD

```
[tv1 z1] = ode45 ('myfun3', [0,0.4],0.9);
plot (tv1, z1,'r -')
grid
title ('Example 3 using Runge-Kutta method')
function z = myfun3(t,y)
z= y.^2;
end
```

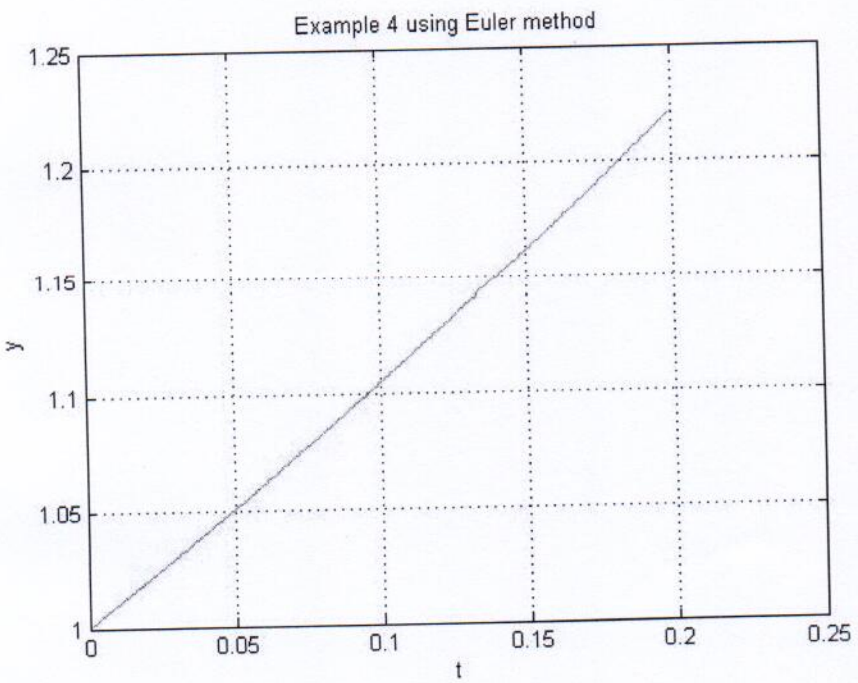


graph 6 showing result to example 3 using Runge-kutta method

Example 4: Solve the initial value problem $y' = y \quad y(0) = 1$ using both Euler and Runge-Kutta methods

MATLAB CODES FOR EULER METHOD

```
t=zeros(201,1);
 y=zeros(201,1);
t(1)=0;
y(1)=1;
for i=1:200
t(i+1)=t(i)+0.001;
y(i+1)=y(i)+0.001*(y(i));
end
plot(t,y,'g -')
grid
title ('Example 4 using Euler method')
```

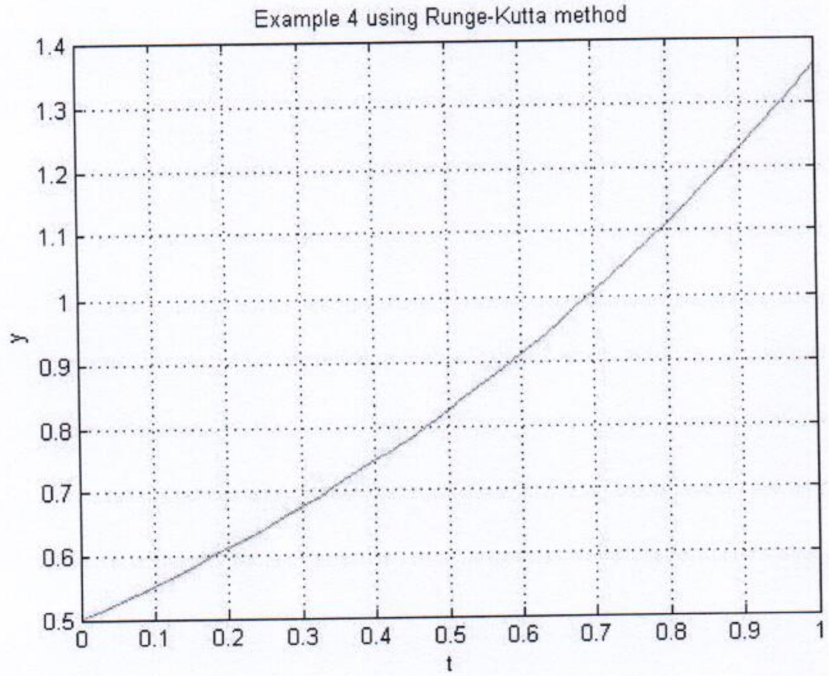


graph 7 showing result to example 4 using Euler method

Solution to Example 4 using Runge-kutta method

MATLAB CODES FOR RUNGE-KUTTA METHOD

```
[tv1 z1] = ode45 ('myfun4', [0,1],0.5);
plot (tv1, z1,'g -')
grid
title ('Example 3 using Runge-Kutta method')
function z = myfun4(t,y)
z= y;
end
```

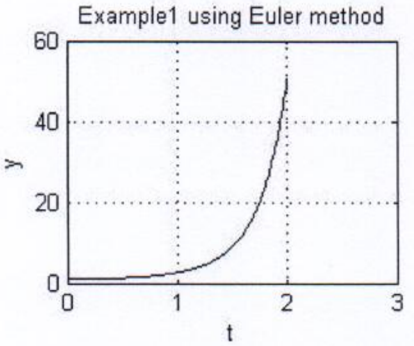


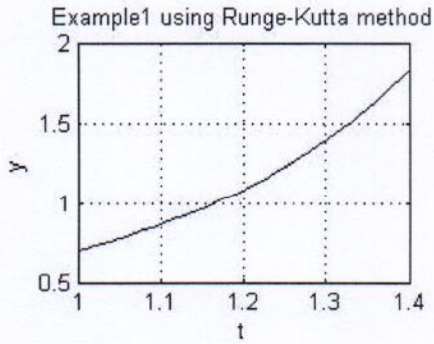
graph 8 showing result to example 4 using Runge-kutta method

CHAPTER FOUR

4. COMPARISON OF THE PERFORMANCE OF THE TWO METHODS

We now proceed to use the solution graphs plotted in the examples in an attempt to compare and contrast the two methods used so far in this work. We start by comparing the result of each method in example 1. The graph below shows the method which best approximates the solution to the initial value problem $\frac{dy}{dx} = 2xy, y(1) = 1$.



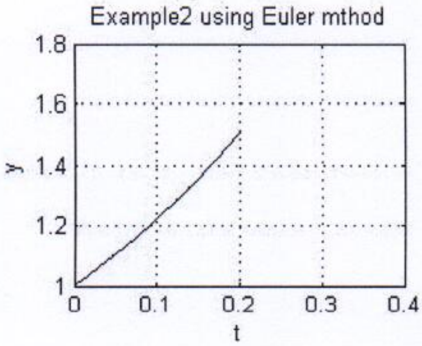


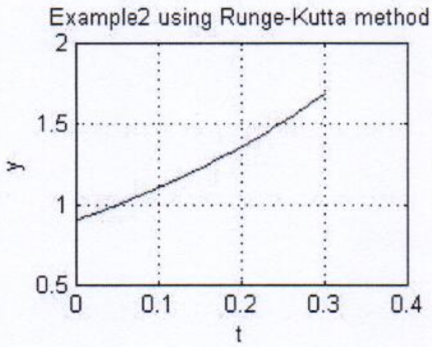
MATLAB CODES

```
[tv1 z1] = ode45 ('myfun2', [1,1.4],0.7);
plot (tv1, z1)
title('Example 1 using runge-kutta');
grid
subplot('221');
t=zeros(201,1);
```

```
y=zeros(201,1);
t(1)=0;
y(1)=1;
for i=1:200
t(i+1)=t(i)+0.01;
y(i+1)=y(i)+0.01*(2*y(i)*t(i));
end
plot(t,y)
grid
subplot('224')
title('Example1 using Euler method');
```

We see from the above graphical solution that the Runge-Kutta method gives a closer approximation to the probelm in question.

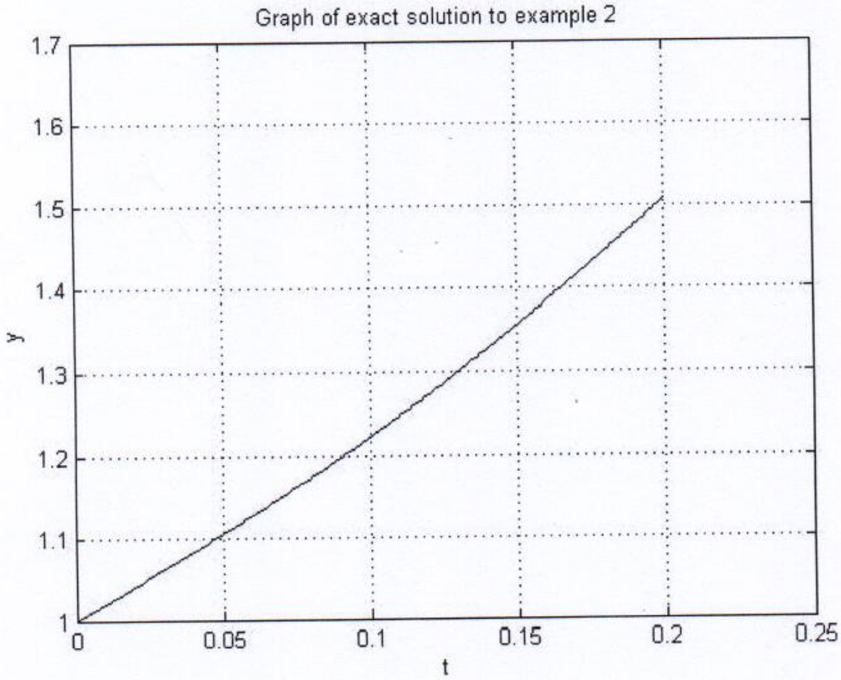




Graphical comparison of the solution to example2

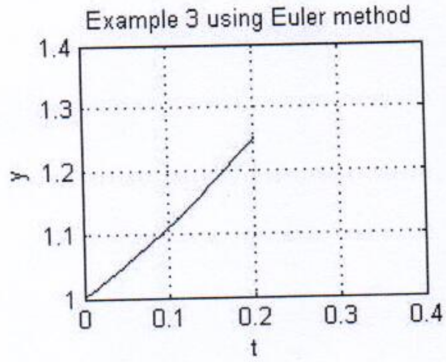
```
[tv1 z1] = ode45('myfun1', [0,0.30],0.9);
plot(tv1, z1,'r -')
title ('Example2 using Runge-Kutta method');
grid
 subplot('221');
t=zeros(201,1);
y=zeros(201,1);
t(1)=0;
y(1)=1;
for i=1:200
 t(i+1)=t(i)+0.001;
y(i+1)=y(i)+0.001*((y(i).^2) + 1);
end
plot(t,y)
grid
title ('Example2 using Euler mthod')
subplot('224')
```

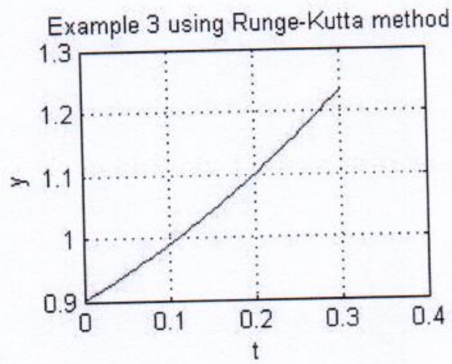
Note that the solution to example 2 using the matlab code $dsolve('Dy - y^2 - 1 = 0','y(0) = 1')$; is $\tan \frac{t+1}{4\pi}$ and the graph of the solution is shown below



We see that the Runge-Kutta gives an almost accurate approximation to the initial value problem.

We also put into consideration the comparison of the solutions in Example 3 from the graph below





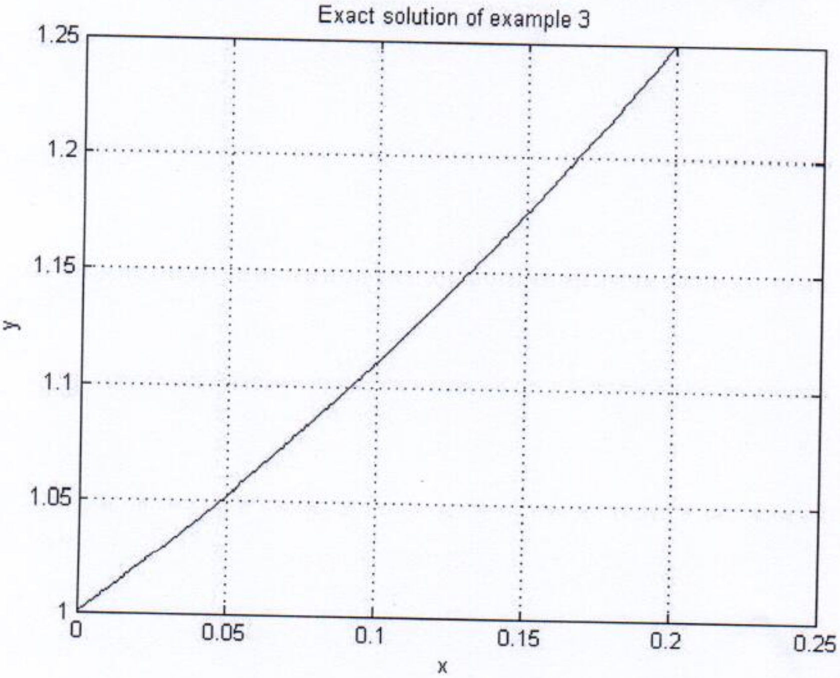
graphical solutions of example

MATLAB CODES

```
tv1 z1= ode45 ('myfun3', [0,0.3],0.9);
plot (tv1, z1,'r -')
grid
title ('Example 3 using Runge-Kutta method')
subplot('221');
t=zeros(201,1);
y=zeros(201,1);
t(1)=0;
y(1)=1;
for i=1:200
t(i+1)=t(i)+0.001;
```

```
y(i+1)=y(i)+0.001*((y(i)^2));
end
plot(t,y);
grid
title ('Example 3 using Euler method')
subplot ('224')
```

We see from the graph of the actual solution below that the Runge-Kutta method is by no means better than the Euler method.



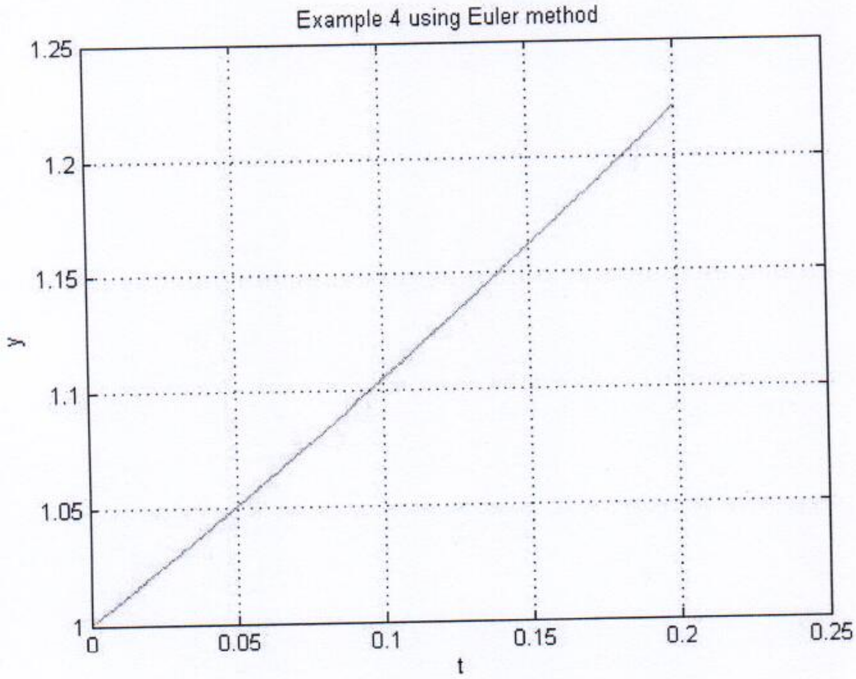
graphical repreentation of the actual solution of example3

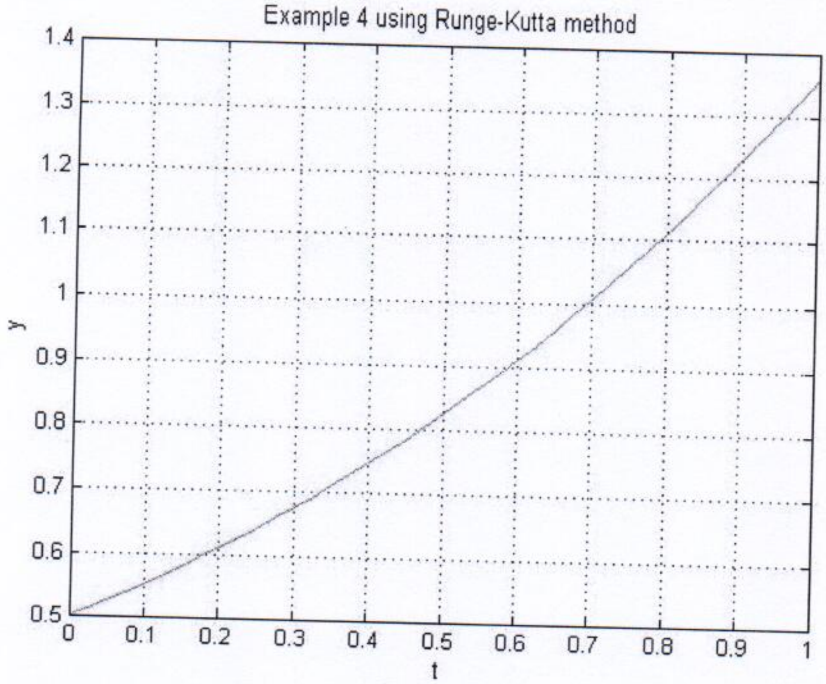
MATLAB CODES

$dsolve('Dy - y^2 = 0',' y(0) = 1');$

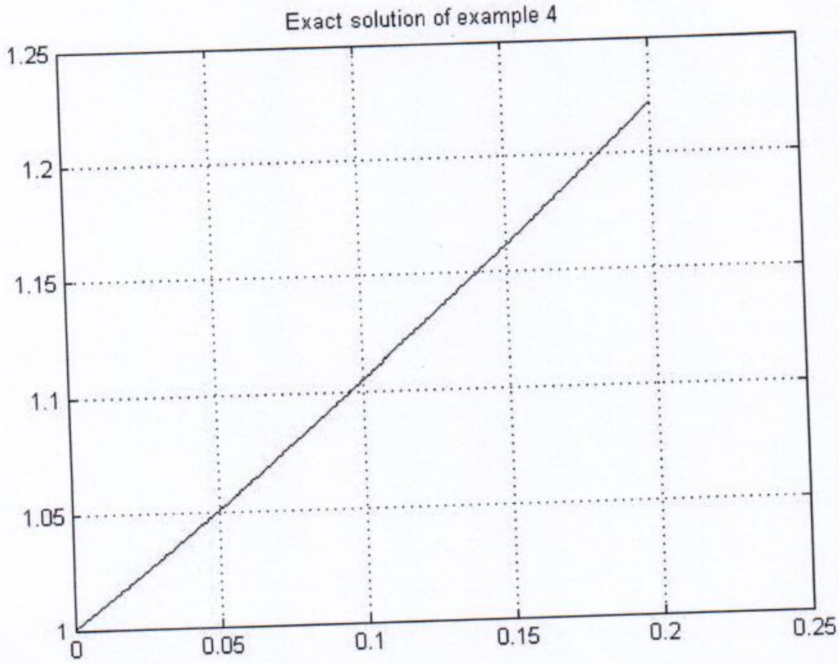
```
plot (t,y);
grid
title ('Exact solution of example 3')
```

Also taking a look at Example4, we see that we can safely conclude that even though the euler method gives a probable solution to the initial value problem, its result can't be fully relied on because of the magnitude of the error. One would rather want to use the Runge-kutta 4th order method as seen from the graphical comparison below





graphical comparison of example4



graphical representation of the actual solution to example4

MATLAB CODES

```
tv1 z1= ode45 ('myfun4', [0,5],0.5);
plot (tv1, z1,'g -');
grid
title ('Example 4 using Runge-Kutta method')
t=zeros(201,1);
y=zeros(201,1);
t(1)=0;
y(1)=1;
for i=1:200
t(i+1)=t(i)+0.001;
y(i+1)=y(i)+0.001*(y(i));
end
plot(t,y,'g -')
grid
title ('Example4 using Euler method')
```

From the solved examples we see that;

- Euler's method has the basic features common to all solution algorithms. The algorithm starts with the given initial value $Y_0 = y(t_0) = \tau$, and then marches forward in time, computing the sequence of approximate solution values $Y_0 = y(x_0)$, $Y_1 \approx y(x_1)$, $Y_2 \approx y(x_2)$, $\ldots$, $Y_p \approx y(x_p)$ in order.

- The higher the number of iterations the higher the error.

In contrast, the 4th order Runge-Kutta method present an "almost accurate" result to the actual solution. It is evident from our result for the 4th order Runge-Kutta method that:

- Better approximations are gotten from higher order of the method as seen in the problem in the last chapter. The best approximation was gotten from the highest order used. A finer approximation will be gotten still if we try using the 5th order Runge-Kutta method.

- Even though the Euler method brought forth the basis for other one-step method numerical solutions to problems, the Runge-Kutta method is a better one-step method to use to get approximation to the numerical solution of the problem as seen in the initial value problem solved.

CONCLUSION

In this work, we've been able to see some of the available computational algorithm one can use to solve differential equations with given initial values. This kind of problems usually arise in other fields like Engineering, Applied Chemistry and even Physics. Therefore, there is a necessity for us to equip ourselves with different tools to tackle problems of this nature.
I would like to state here that the methods I have stated or used here are not the only methods available to solve these kind of problems as we all know. These methods are only but a few one-step algorithms/methods out of the numerous available methods. We have other one-step mehods like the Heun's method, 5th order Runge-Kutta methods and other multi-step methods like Adams method, Adams-Bashforth methods, Adams-Moulton methods just to mention a few. Subsequent works of this type can compare or contrast the effectiveness of these methods.