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Noise-induced outer synchronization between two different complex dynamical networks

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Abstract In this paper, based on the theory of stochastic differential equations, we study the outer synchronization between two different complex dynamical networks with noise coupling. The theoretical result shows that two different complex networks can achieve generalized outer synchronization only with white-noise-based coupling. Numerical examples further verify the effectiveness and feasibility of the theoretical results. Numerical evidence shows that the synchronization rate is proportional to the noise intensity.

Keywords Complex network · Synchronization · Noise

1 Introduction

Recently, synchronization of complex networks has been extensively investigated [1–8]. It should be noted that most previous studies mainly focused on the "inner synchronization" [9–13], which is concerned with the synchronization among the nodes within a network. Except for inner synchronization, we can also observe other kinds of synchronization behavior for complex networks, such as outer synchronization between two coupled networks, which means that the corresponding nodes of coupled networks will achieve synchronization regardless of synchronization of the inner networks. Example of the outer synchronization of complex networks includes the balance between predator–prey networks in ecological communities [14]. Compared with the rich work with respect to the inner synchronization of complex networks, the results on the outer synchronization are less numerous.

In the early papers on the outer synchronization, it is usually assumed that the corresponding nodes in two networks manifest completely the same dynamics [15–18]. However, in reality, nodes in different networks usually have different dynamics, while the two networks may still behave in a synchronous way. So, the study of the outer synchronization between two networks with different dynamical behaviors is very important to the perspective of control theory and practical applications. Some theoretical results for the outer synchronization between two complex networks with different dynamics have been obtained [19–25]. In Ref. [19], the generalized outer synchronization between two completely different complex networks was investigated. The concept of mixed outer synchronization was introduced in Ref. [20]. Wu et al. [21] investi-

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gated the generalized outer synchronization in master–slave networks with the open-plus-closed-loop control technique. More recently, the finite-time generalized outer synchronization between two different complex networks was investigated [25].

Since noise is ubiquitous in the real systems, the synchronization of coupled systems or networks is unavoidably affected by different kinds of noise. Therefore, the effect of noise on synchronization has been well studied by many scientists [26–31]. Noise is commonly regarded as a persistent disturbance which usually inhibits synchronization. However, recent researchers have reported that noise could also play a constructive role in nonlinear systems [28–31]. Lin et al. [28] analytically showed that chaos synchronization between two unidirectionally coupled chaotic systems could be induced by noise when noise presents in coupling term, and proposed a good analytical method to analyze this phenomenon. By using Lin's method, Xiao et al. investigated the effect of noise on the synchronization of two bidirectionally coupled chaotic systems [29]. Although noise-induced synchronization of uncoupled oscillators or coupled-oscillator network with small size has been extensively studied for both periodic and chaotic oscillators [28, 29], works on the constructive role of noise in the synchronization of networks with a large population of coupled oscillators are few. Negail et al. [30] showed that common noise can induce the synchronization of a large population of globally coupled nonidentical oscillators. More recently, Xiao et al. showed that noise plays a constructive role in the synchronization of globally coupled dynamical network [31]. Thus, it becomes interesting to ask such questions as the following: "Can two completely different complex networks achieve certain synchronization only with noise coupling?" Or: "Besides the numerical evidence, are there any analytical arguments illustrating such constructive effects of noise?"

Motivated by the above discussion, we devote ourselves to focusing on the effect of noise on the outer synchronization between two completely different complex networks. In our study, each network can be undirected or directed, connected or disconnected, and both networks have different dynamics and topologies. Based on the stability theory of stochastic differential equations, we analytically show that two networks can realize generalized outer synchronization only with white-noise-based coupling.

The rest of this paper is organized as follows. In Sect. 2, the network models and some useful preliminaries are given. Based on the stability theory of stochastic differential equations, sufficient conditions for the generalized outer synchronization are derived analytically in Sect. 3. In Sect. 4, two numerical examples are given to show the effectiveness of the theoretical results. Finally, some conclusions are drawn in Sect. 5.

Notation Throughout this paper, unless specified, we let $\|\cdot\|$ be Euclidean norm, I be an identity matrix of suitable dimensions. If A is a vector or matrix, its transpose is denoted by A^T. If A is a symmetric matrix, $\lambda_{\max}(A)$ denotes its largest eigenvalue.

2 Network modeling and preliminaries

Consider a general complex network consisting of N dynamical nodes with linear couplings, which is described by

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{N} a_{ij} P x_j, \quad i = 1, 2, \ldots, N, \tag{1}$$

where $x_i = (x_{i1}, \ldots, x_{in})^T \in R^n$ is the state vector of the ith node, $f : R^n \to R^n$ is a continuously differentiable nonlinear vector function governing the evolution of the ith isolated node x_i; $P \in R^{n\times n}$ is the inner connection matrix between two connected nodes; $A = (a_{ij})_{N\times N}$ represents the coupling configurations of the network, whose entries a_{ij} are defined as follows: if there is a link from node j to node i $(i \neq j)$ then set $a_{ij} > 0$, otherwise $a_{ij} = 0$ $(i \neq j)$. The diagonal elements of matrix A are defined as

$$a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}, \quad i = 1, 2, \ldots, N.$$

To realize the generalized outer synchronization between two coupled complex networks, we refer to model (1) as the drive network, and the response network is given by the following equations:

$$\dot{y}_i = g(y_i) + \sum_{j=1}^{N} b_{ij} Q y_j + u_i, \quad i = 1, 2, \ldots, N, \tag{2}$$

where $y_i = (y_{i1}, \ldots, y_{im})^T \in R^m$ is the state vector of node i, $g : R^m \to R^m$ is a continuously differentiable nonlinear vector function governing the evolution of the ith isolated node y_i. $Q \in R^{m\times m}$ is the inner connection matrices, $B = (b_{ij})_{N\times N}$ is the coupling configuration matrix, which has the same meaning as that of matrix A. u_i $(i = 1, 2, \ldots, N)$ are the controllers defined as follows:

$$u_i = \mathcal{D}\Phi_i(x_i)\dot{x}_i - g\big(\Phi_i(x_i)\big) - \sum_{j=1}^{N} b_{ij} Q\Phi_j(x_j) + h_i\big(y_i - \Phi_i(x_i)\big)\dot{W}(t), \tag{3}$$

where $\mathcal{D}\Phi_i(x_i)$ is the Jordan matrix of the vector function $\Phi_i(x_i)$, and $h_i : R^n \to R^{m\times s}$ is called the noisy intensity matrix function; $W(t) = (w_1, \ldots, w_s)^T$ is an s-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$. $\dot{W}(t) = [\xi_1(t), \ldots, \xi_s(t)]$ is an s-dimensional white noise vector with statistical properties $\langle \xi_i(t)\rangle = 0$ and $\langle \xi_i(t), \xi_j(t')\rangle = \delta_{ij}\delta(t - t')$ $(i, j = 1, 2, \ldots, m)$.

Remark 1 In this paper, the configuration matrices A and B of networks (1) and (2) are not assumed to be symmetric or irreducible, which means that networks (1) and (2) can be undirected or directed networks, and they may also contain isolated nodes and clusters.

Since a linear controller has the simplest structure and can be implemented more easily and efficiently, therefore, for a better presentation, we set $h_i(y_i - \Phi_i(x_i)) = \sigma_0 \cdot (y_i - \Phi_i(x_i))$: being coincident with this $\dot{W}(t)$ reduces to a one-dimensional white noise. The extension to the case of s-dimensional white noise is straightforward; we will only show a final result for $s = 1$ in this paper. For this case, the coupling strength $\sigma_0\dot{W}(t)$ actually becomes white noise with strength σ_0. This is in accordance with the many realistic phenomena that the coupling strengths of real oscillator networks and biological systems are time varying due to the perturbation of external or intrinsic noise.

In the case of one-dimensional white noise, the focus of this paper, the controllers in Eq. (3) can be rewritten as

$$u_i = \mathcal{D}\Phi_i(x_i)\dot{x}_i - g\big(\Phi_i(x_i)\big) - \sum_{j=1}^{N} b_{ij} Q\Phi_j(x_j) + \sigma_0\big(y_i - \Phi_i(x_i)\big)\dot{W}(t), \quad i = 1, 2, \ldots, N. \tag{4}$$

Next, we first give the definition of outer synchronization between two networks, followed by an assumption, which are required throughout this paper.

Definition 1 Networks (1) and (2) are said to achieve outer synchronization with probability one if, for any initial state $x_i(0)$, $y_i(0)$,

$$P\Big\{\lim_{t\to\infty} \big\| y_i\big(t, y_i(0)\big) - x_i\big(t, x_i(0)\big)\big\| = 0\Big\} = 1, \quad i = 1, 2, \ldots, N. \tag{5}$$

Let $\Phi_i : R^n \to R^m$ be continuously differentiable function. Networks (1) and (2) are said to achieve generalized outer synchronization with probability one if, for any initial state $x_i(0)$, $y_i(0)$,

$$P\Big\{\lim_{t\to\infty} \big\| y_i\big(t, y_i(0)\big) - \Phi_i\big(x_i\big(t, x_i(0)\big)\big)\big\| = 0\Big\} = 1, \quad i = 1, 2, \ldots, N. \tag{6}$$

Assumption 1 *For function $g(x)$ there exists a positive constant L such that*

$$(x - y)^T\big(g(x) - g(y)\big) \leq (x - y)^T L(x - y),$$

for all $x, y \in R^m$.

Remark 2 The above condition in Assumption 1 is usually called global Lipschitz condition, and L is called Lipschitz constant. It is easy to check that some well-known chaotic systems, such as Chuan's circuit [32] and Rössler-like system [33], satisfy Assumption 1.

3 Main results

In what follows, based on the stability theory of stochastic differential equations, it is shown that generalized outer synchronization between networks (1) and (2) could be achieved with probability one just through adjusting the intensity of white noise. The main results are given in the following theorem.

Theorem 1 *Suppose that Assumption 1 holds. If there exists a sufficiently large noise strength σ_0 such that*

$$\sigma_0^2 > 2\big(\lambda_{\max}\big(\mathcal{Q}^s\big) + L\big),$$

where $\mathcal{Q} = B \otimes Q$, $\mathcal{Q}^s = \frac{\mathcal{Q}+\mathcal{Q}^T}{2}$, then, under controllers (4), networks (1) and (2) can reach generalized outer synchronization with probability one.

Proof Define $e_i = y_i - \Phi_i(x_i)$ to be the error state between networks (1) and (2); then one gets the following error system:

$$\dot{e}_i = g(y_i) - g\big(\Phi_i(x_i)\big) + \sum_{j=1}^{N} b_{ij} Q e_j + \sigma_0 \cdot e_i \dot{W}(t), \quad i = 1, 2, \ldots, N. \tag{7}$$

Let $e = (e_1^T, e_2^T, \ldots, e_N^T)^T$; then we can rewrite system (7) in a compact form as follows:

$$\dot{e} = G(X, Y) + B \otimes Qe + \sigma_0 \cdot e\dot{W}(t), \tag{8}$$

where $G(X, Y) = (g(y_1) - g(\Phi_1(x_1)), \ldots, g(y_N) - g(\Phi_N(x_N)))^T$.

Applying the theory of stochastic differential equation (see Ref. [34, p. 51]), the error dynamics (8) poses a unique global solution on $t \geq 0$, denoted by $e(t, e_0)$, for any initial data $e_0 \in R^{mN}$. Obviously, $e(t, 0) \equiv 0$ is a trivial solution of the error system (8). The generalized outer synchronization between networks (1) and (2) is achieved in a statistical sense of probability one, if this trivial solution is globally almost surely asymptotically stable , i.e. $\lim_{t\to\infty} \|e\| = 0$ with probability one.

In what follows, we will give sufficient conditions for the generalized outer synchronization between networks (1) and (2) with probability one. To this end, we introduce the following function:

$$V(e) = \frac{1}{2} \log e^T e. \tag{9}$$

By the Itô formula to (9), along the solution of system (8), we have

$$V(e) = V\big(e(t_0)\big) + \int_{t_0}^{t} \mathcal{L}\big[V(e)\big] \mathrm{d}s + \int_{t_0}^{t} V_e(e)\sigma_0 e \mathrm{d}W(s), \tag{10}$$

where

$$\mathcal{L}\big[V(e)\big] = V_e(e)\big(G(X, Y) + B \otimes Qe\big) + \frac{1}{2}\mathrm{trace}\big(\sigma_0 e^T V_{ee}(e)\sigma_0 e\big), \tag{11}$$

and

$$V_e(e) = \frac{e^T}{e^T e}, \qquad V_{ee}(e) = \frac{I}{e^T e} - 2\frac{ee^T}{(e^T e)^2}.$$

From Assumption 1, we get

$$\begin{aligned}\mathcal{L}\big[V(e)\big] &= \frac{1}{e^T e}\big(e^T G(X, Y) + e^T B \otimes Qe\big) + \frac{1}{2}\mathrm{trace}\left(\frac{\sigma_0^2 e^T e}{e^T e} - 2\frac{\sigma_0^2 (e^T e)^2}{(e^T e)^2}\right)\\ &\leq \frac{1}{e^T e}\big(Le^T e + \lambda_{\max}\big(\mathcal{Q}^s\big)e^T e\big) - \frac{\sigma_0^2}{2}\\ &= \lambda_{\max}\big(\mathcal{Q}^s\big) + L - \frac{\sigma_0^2}{2}.\end{aligned}$$

Thus, we have

$$V\big(e(t)\big) \leq V\big(e(t_0)\big) + \left(\lambda_{\max}\big(\mathcal{Q}^s\big) + L - \frac{\sigma_0^2}{2}\right)(t - t_0) + \mathcal{M}(t), \tag{12}$$

where

$$\mathcal{M}(t) = \int_{t_0}^{t} V_e(e)\sigma_0 e \mathrm{d}W(s) \tag{13}$$

$$= \int_{t_0}^{t} \frac{e^T \sigma_0 e}{e^T e} \mathrm{d}W(s). \tag{14}$$

It is easy to see that $\mathcal{M}(t_0) = 0$, and the quadratic variation is

$$\big[\mathcal{M}(t), \mathcal{M}(t)\big] = \int_{t_0}^{t} \frac{e^T \sigma_0 e \sigma_0 e^T e}{(e^T e)^2} \mathrm{d}s \tag{15}$$

$$= \sigma_0^2 (t - t_0). \tag{16}$$

Therefore, according to the strong law of large numbers (see Ref. [34, p. 12]), one has that

$$\lim_{t\to\infty} \frac{\mathcal{M}(t)}{t} = 0, \quad \text{a.s.} \tag{17}$$

Combing Eqs. (10)–(17), we have an estimation of $V(t)$ as follows:

$$\limsup_{t\to\infty} \left[\frac{\log(e^T e)}{2t}\right] \leq \lambda_{\max}\big(\mathcal{Q}^s\big) + L - \frac{\sigma_0^2}{2}, \quad \text{a.s.} \tag{18}$$

Therefore, if

$$\sigma_0^2 > \sigma_c^2 \doteq 2\big(\lambda_{\max}\big(\mathcal{Q}^s\big) + L\big), \tag{19}$$

then the trivial solution of (8) is almost surely exponentially stable. This means that generalized outer synchronization between networks (1) and (2) could be achieved with probability one provided that the strength of white noise is large enough. The proof is completed. □

Remark 3 The analytical result in Theorem 1 provides us a sufficient condition for the generalized outer synchronization between noise-coupled networks (1) and (2) in a statical sense, according to which as long as the intensity of the coupling noise satisfies $\sigma_0^2 > 2(\lambda_{\max}(\mathcal{Q}^s) + L)$, the response network can synchronize the drive network with a time shift. In fact, the preceding mathematical proof shows that the convergence rate of the synchronization can be estimated from (18) and the generalized outer synchronization is achieved if $\lambda_{\max}(\mathcal{Q}^s) + L - \sigma_0^2/2$ is less than zero. In the case of noise strength $\sigma_0 = 0$, the damping rate (18) becomes $\lambda_{\max}(\mathcal{Q}^s) + L$. Based on the classical Lyapunov direct method, one can easily prove that the generalized outer synchronization between the noiseless networks (1) and (2) could be achieved if $\lambda_{\max}(\mathcal{Q}^s) + L < 0$. Moreover, when $\lambda_{\max}(\mathcal{Q}^s) + L$ is not less than zero, the damping rate (18) shows that the generalized outer synchronization can be achieved by adding the intensity of the coupling noise. Therefore, we can see that the noise may indeed have beneficial significance and impact on the synchronization.

Remark 4 From (18) we can see that the convergence rate of the synchronization is exponential and the larger the intensity of the noise, the faster the convergence speed. Moreover, the threshold of the intensity of coupled white noise can be estimated by the sufficient condition (19). However, this kind of sufficient condition might give an overestimated threshold of the intensity of noise. The simulation results in next section will show that outer synchronization could be caused by a white noise with much lesser value of intensity than the threshold estimated by the inequality (19).

From Definition 1 we know that the complete outer synchronization is a kind of special case of the generalized outer synchronization if $\Phi_i(x) = x$, $i = 1, 2, \ldots, N$. Based on Theorem 1, we can easily derive the following corollaries:

Corollary 1 *Let Assumption 1 hold. If networks (1) and (2) have identical dynamics, namely, $f = g$ and $\sigma_0^2 > \sigma_c^2$, then the two networks can achieve complete outer synchronization with probability one under the following control scheme*:

$$u_i = \sigma_0(y_i - x_i)\dot{W}(t) + \sum_{j=1}^{N}(a_{ij}P - b_{ij}Q)x_j, \quad i = 1, 2, \ldots, N. \tag{20}$$

Corollary 2 *Let Assumption 1 hold. If networks (1) and (2) have the same topological structures and uniform inner-coupling matrices, i.e., $A = B$, $P = Q$, and also have identical node dynamics, namely, $f = g$, then the two networks can achieve complete outer synchronization with probability one under the following control scheme*:

$$u_i = \sigma_0(y_i - x_i)\dot{W}(t), \quad i = 1, 2, \ldots, N, \tag{21}$$

where $\sigma_0^2 > \sigma_c^2$, and σ_c^2 is defined as that in (19).

4 Numerical simulations

In this section, illustrative examples are given to verify the effectiveness of the theoretical criteria obtained in the preceding section. In the numerical simulations, we consider several networks with ten nodes. For brevity, we always set $P = Q = I$. The initial conditions of the drive network and the response network are randomly taken from the interval $[-1, 1]$. The total synchronization error $E(t) = \|e\|$ is used to measure the evolution process.

Example 1 In this example, we use the Lorenz system to describe the node dynamics of the driving network and take the Rössler-like system as the node dynamics of the response network.

The Lorenz system can be described as

$$\dot{x}_i = f(x_i) = \begin{pmatrix} -a & a & 0 \\ c & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \begin{pmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{pmatrix} + \begin{pmatrix} 0 \\ -x_{i1}x_{i3} \\ x_{i1}x_{i2} \end{pmatrix}, \tag{22}$$

where $x_i = (x_{i1}, x_{i2}, x_{i3})^T \in R^3$ is the state vector. System (22) has a double-scrolling chaotic attractor when $a = 10$, $b = 8/3$, $c = 28$.

The Rössler-like system can be described as [33]:

$$\dot{y}_i = g(y_i) = \alpha \begin{pmatrix} -\Gamma & -\beta & -\lambda \\ 1 & \gamma & 0 \\ 0 & 0 & -\mu \end{pmatrix} \begin{pmatrix} y_{i1} \\ y_{i2} \\ y_{i3} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \alpha\mu\psi(y_{i1}) \end{pmatrix} \tag{23}$$

where $y = (y_{i1}, y_{i2}, y_{i3})^T \in R^3$ is the state vector,

$$\psi(s) = \begin{cases} 0, & s < 2.56; \\ \xi(s - 2.56), & s \geq 2.56. \end{cases} \tag{24}$$

The Rössler-like system has a chaotic attractor when $\alpha = 0.03$, $\beta = 1.5$, $\gamma = 0.2$, $\mu = 1.5$, $\lambda = 0.75$, $\xi = 21.43$ and $\Gamma = 0.075$. And it is easy to compute that the Rössler-like system satisfies the Assumption 1 with $L = 0.4926$.

The map Φ_i is defined as

$$\Phi_i(x_i) = \left(x_{i1} + x_{i3}, 2x_{i2} + 1, \frac{1}{2}x_{i3}^2 - x_{i1}\right)^T,$$

$$i = 1, 2, \ldots, N.$$

Then

$$\mathcal{D}\Phi_i(x_i) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & x_{i3} \end{pmatrix}.$$

According to the controllers in (4) and Eq. (23), we can present the response network as follows:

$$\begin{aligned}
\dot{y}_{i1} &= -\alpha\Gamma y_{i1} - \alpha\beta y_{i2} - \alpha\lambda y_{i3} + \alpha\beta \\
&\quad + (\alpha\Gamma - \alpha\lambda - a)x_{i1} + (2\alpha\beta + a)x_{i2} \\
&\quad + (\alpha\Gamma - b)x_{i3} + 0.5\alpha\lambda x_{i3}^2 + x_{i1}x_{i2} \\
&\quad + \sum_{j=1}^{10} a_{ij}(x_{j1} + x_{j3}) + \sum_{j=1}^{10} b_{ij}e_{j1} \\
&\quad + \sigma_0 e_{i1}\dot{W}(t), \\
\dot{y}_{i2} &= \alpha y_{i1} + \alpha\gamma y_{i2} + (2c - \alpha)x_{i1} - (2\alpha\gamma + 2)x_{i2} \\
&\quad - \alpha x_{i3} - 2x_{i1}x_{i3} - \alpha\gamma + 2\sum_{j=1}^{10} a_{ij}x_{j2} \\
&\quad + \sum_{j=1}^{10} b_{ij}e_{j2} + \sigma_0 e_{i2}\dot{W}(t), \\
\dot{y}_{i3} &= -\alpha\mu\big(y_{i3} - \psi(y_{i1}) + \psi(x_{i1} + x_{i3})\big) \\
&\quad + (a - \alpha\mu)x_{i1} - ax_{i2} + (0.5\alpha\mu - b)x_{i3}^2 \\
&\quad + x_{i1}x_{i2}x_{i3} + x_{i3}\sum_{j=1}^{10} a_{ij}x_{j3} - \sum_{j=1}^{10} a_{ij}x_{j1} \\
&\quad + \sum_{j=1}^{10} b_{ij}e_{j3} + \sigma_0 e_{i3}\dot{W}(t).
\end{aligned} \tag{25}$$

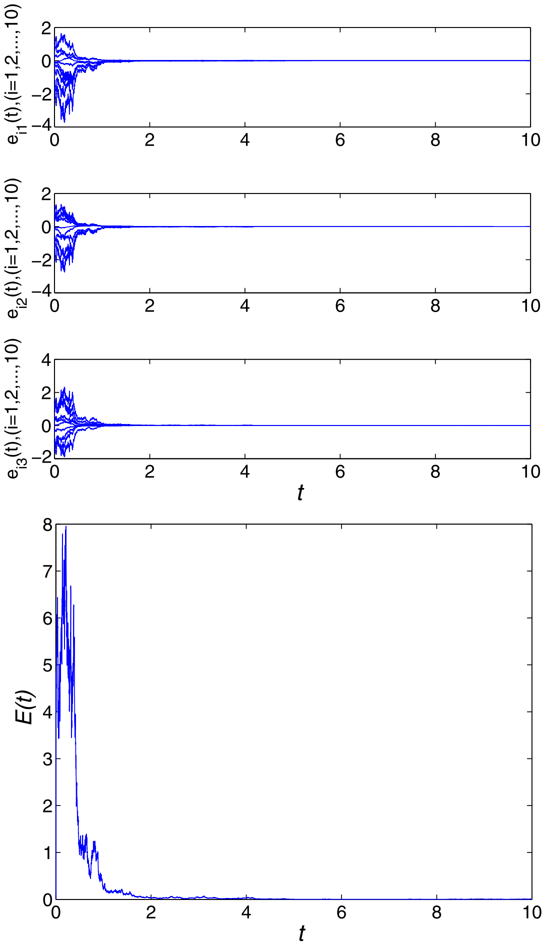
The configuration matrix for the drive network is given as follows:

$$A = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -3 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & -3 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & -3 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & -3 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & -2 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2
\end{pmatrix}.$$

The configuration matrix for the response network is given as follows:

$$B = \begin{pmatrix}
-3 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & -3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}.$$

It is easy to compute that $\lambda_{\max}(\mathcal{Q}^s) = 0.3896$ and $\sigma_c = 1.33$. According to Theorem 1, networks (1) and (2) can reach generalized outer synchronization if the noise intensity σ_0 is larger than σ_c. Taking $\sigma_0 = 1.4$ ($>\sigma_c$), we simulate the evolution of the networks according to the controllers defined in (4).



(a)

(b)

Fig. 1 Trajectories of the synchronization error (**a**) and the total synchronization error (**b**) between networks (1) and (2) with $\sigma_0 = 1.4$

The synchronization errors $e_{ij}(t)$ $(i = 1, 2, \dots, 10;$ $j = 1, 2, 3)$ and the total synchronization error $E(t)$ are shown in Figs. 1(a) and (b), which indicates that the generalized outer synchronization between two networks is achieved. To study the effect of noise on the convergence speed, we simulate the evolution of two networks according to the protocol defined in Eq. (4) with various values of σ_0. The results are exhibited in Fig. 2. One can see that the synchronization speed increases with the noise strength, which is consistent with the analysis of Remark 3.

Example 2 In this example, we take the Lorenz system as the node dynamics of the drive network (1) and Chua's circuit as the node dynamics of the response network (2).

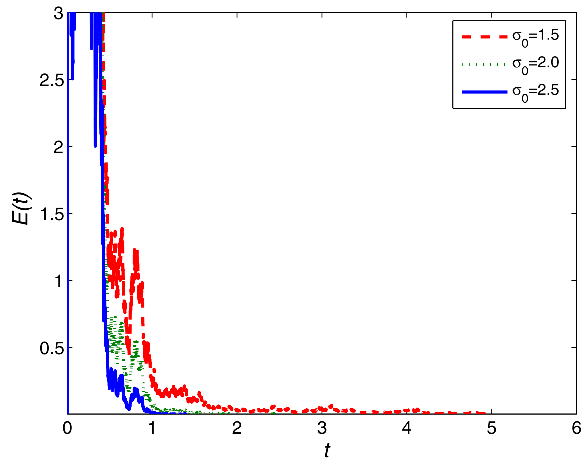


Fig. 2 Time evolutions of total synchronization error $E(t)$ with noise strength $\sigma_0 = 1.5, 2.0, 2.5$

The Chua's circuit can be depicted by three-dimensional differential equations [32]:

$$\dot{y}_i = g(y_i) = \begin{pmatrix} -p - pb & p & 0 \\ 1 & -1 & 1 \\ 0 & -q & 0 \end{pmatrix} \begin{pmatrix} y_{i1} \\ y_{i2} \\ y_{i3} \end{pmatrix} + \begin{pmatrix} \omega(y_{i1}) \\ 0 \\ 0 \end{pmatrix} \tag{26}$$

where $y_i = (y_{i1}, y_{i2}, y_{i3})^T \in R^3$ is the state vector, $\omega(x_1) = 0.5p(b - a)(|x_1 + 1| - |x_1 - 1|)$. In all of the simulations, we always choose the system parameters of the Chua's circuit as $p = 10, q = 14.87$, $a = -1.27, b = -0.68$, which causes the Chua's circuit to exhibit a double-scroll chaotic attractor. It is easy to verify that the Chua's circuit satisfies Assumption 1 with $L = 13.76$.

The map ϕ_i is defined as

$$\phi_i(x_i) = (-x_{i1}, 2x_{i2}, -3x_{i3})^T, \quad i = 1, 2, \dots, N.$$

Then

$$\mathcal{D}\phi_i(x_i) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

According to the controllers in (4) and Eq. (26), we can present the response network as follows:

$$\dot{y}_{i1} = -(p + pb)y_{i1} + py_{i2} + \omega(y_{i1}) + \omega(x_{i1}) + (a - p - pb)x_{i1} - (a + 2p)x_{i2}$$

$$-\sum_{j=1}^{10} a_{ij}x_{j1} + \sum_{j=1}^{10} b_{ij}e_{j1} + \sigma_0 e_{i1}\dot{W}(t),$$

$$\dot{y}_{i2} = y_{i1} - y_{i2} + y_{i3} + 2cx_{i1} + 3x_{i3} - 2x_{i1}x_{i3} \quad (27)$$

$$+2\sum_{j=1}^{10} a_{ij}x_{j2} + \sum_{j=1}^{10} b_{ij}e_{j2} + \sigma_0 e_{i2}\dot{W}(t),$$

$$\dot{y}_{i3} = -qy_{i2} + 2qx_{i2} + 3bx_{i3} - 3x_{i1}x_{i2}$$

$$-3\sum_{j=1}^{10} a_{ij}x_{j3} + \sum_{j=1}^{10} b_{ij}e_{j3} + \sigma_0 e_{i3}\dot{W}(t).$$

In the numerical simulations, the configuration matrices for the networks (1) and (2) are taken as those in Example 1. It is easy to compute that $\sigma_c = 5.32$. One should note that the theoretical criterion in Theorem 1 is just a sufficient condition, and networks (1) and (2) will synchronize when σ_0 exceeds a value which is less than σ_c. To obtain a critical value of the noise strength σ_0 to achieve outer synchronization, we continuously increase the noise strength σ_0 from $\sigma_0 = 0.1$, in steps of 0.1. When $\sigma_0 < 3.2$ no synchronous phenomenon is observed. The critical value of the noise strength σ_0 is 3.2. Taking $\sigma_0 = 3.2$ ($< \sigma_c$), we simulate the evolution of the networks according to the controllers defined in (4). The numerical results in Figs. 3(a) and (b) indicate that the generalized outer synchronization between two networks is reached, and the simulations match the theoretical results perfectly. Figure 4 shows that the synchronization speed increases with the noise strength. These figures show that numerical results are consistent with the theoretical results acquired in Sect. 3.

5 Conclusions

In this paper, based on the theory of stochastic differential equation, the constructive role of noise in the outer synchronization between two different complex networks has been investigated analytically and numerically. Theoretical results show that generalized outer synchronization between two different complex networks could be achieved by increasing the strength of noise. This shows that the noise really has a positive effect on the synchronization. Two representative examples are provided to show the effectiveness and

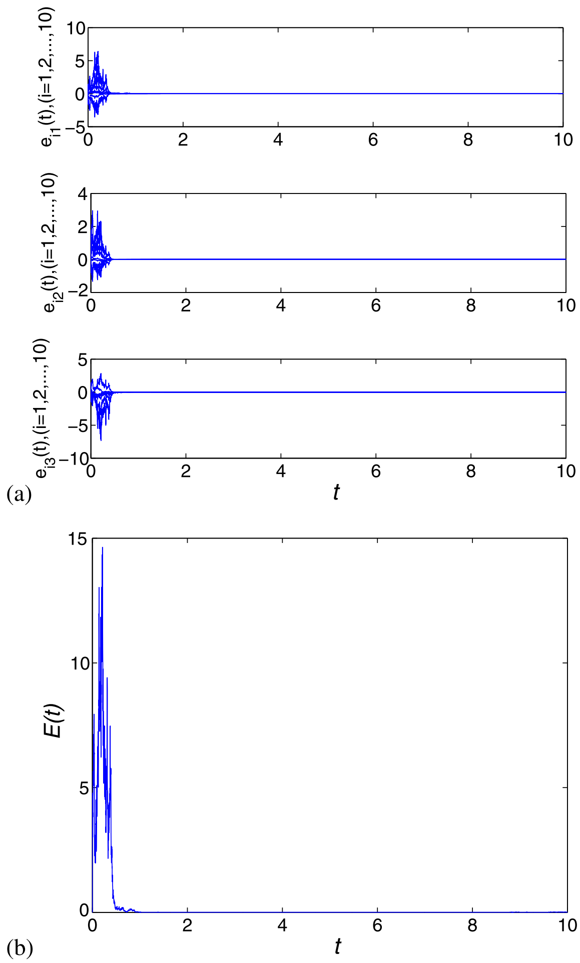


Fig. 3 Trajectories of the synchronization error (**a**) and the total synchronization error (**b**) between networks (1) and (2) with $\sigma_0 = 3.2$

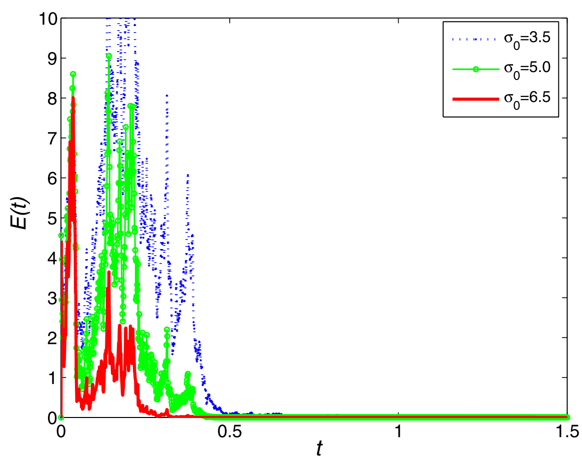


Fig. 4 Time evolutions of total synchronization error $E(t)$ with noise strength $\sigma_0 = 3.5, 5.0, 6.5$

feasibility of the theoretical results. The simulation results show that the synchronization speed is proportional to the noise strength. The results in this paper will be helpful in understanding the role of noise in network synchronization. In addition, except for unavoidable noise, time delays due to the finite information transmission and processing speeds among the network nodes is another factor which may affect the behavior of dynamics between coupled networks. The present study does not consider the effect of time delays; however, research is being pursued in this direction.

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