A CONCISE APPROACH TO ELECTROMAGNETIC FIELDS AND WAVES

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First Edition (2015)

ISBN: 978-2528-75-1

Published by:

VITAL PUBLISHING & PRINTING COMPANY

ADO-EKITI

@ 08033229194 and 08027330612

ACKNOWLEDGEMENT

We are grateful to the following: Prof. Engr. B.A Alabadan, the Dean of Faculty of Engineering, Federal University Oye-Ekiti,; Dr. Engr. A.M. Zungeru, the immediate past Head of the Department of Electrical and Electronics Engineering, Federal University Oye-Ekiti, and the entire staff of the Department of Electrical and Electronics Engineering, Federal University Oye-Ekiti, for their encouragements and motivations.

Special thanks go Dr. Engr. M.B Muazu, the Head of Department of Electrical and Computer Engineering, Ahmadu Bello University, Zaria and the entire academic and non-academic staff of the Department for their encouragements and motivations to write this book.

We are indebted to all the students in Ahmadu Bello University, Zaria [1999-2014] and Federal University Oye-Ekiti [2011-2014] for their strong motivations to bring this book.

We also appreciate the contribution of Prof. Engr. B.A. Adegboye, the Head of Department of Electrical and Electronics Engineering, Federal University, Minna.

We sincerely appreciate the moral and spiritual contribution of the new Vice-Chancellor of Ekiti State University, Ado Ekiti, in the person of Prof. S.O. Bandele.

We are grateful to our beloved wives Mrs. Fatima Jumai Akinsanmi and Mrs. Peace C. Ijemaru for inspiring us for this project.

Lastly, we express our heartfelt thanks to the publisher Vital A. Ventures for publishing this book in such a beautiful get-up and well in time.

PREFACE

*As the name implies, **"A Concise Approach to Electromagnetic Fields and Waves"** is an outgrowth of the lecture notes prepared for Electromagnetic Fields and Waves I & II. Intended primarily as a text book for students at the advanced undergraduate or beginning graduate level, it is expected that the book will be useful for research workers too. It provides a thorough treatment of the theory of electrodynamics, mainly from a classical field theoretical point of view, and includes such things as electrostatics, electromagnetic waves and their propagation in vacuum and in media.*

With the parameters and terms that made up the text clearly defined, this book is based on classical electromagnetism that deals with the study of static charges and their electric fields, the magnetic field developed by motional charges and the electromagnetic wave that treated the dynamism of electron in rapidly time-varying condition.

*The book – **"A Concise Approach to Electromagnetic Fields and Waves"** consists of eight chapters. Chapter one is an introduction to electromagnetic fields and waves. This chapter x-rays the basic concepts about electromagnetics, the discovery of electromagnetic field, Maxwell's equations, electromagnetic spectrum and the fundamental characteristics of EM waves. Chapter two looks into a review of vector analysis, which includes vector algebra and vector calculus, while chapter three presents Electrostatics and its applications as well as charge distribution in a conductor, Coulomb's law, electric field intensity, and electric potential. In chapter four, there is a review of EM laws in the integral form, which houses Gauss's law, Ampere's law, Faraday's law, magnetic field in and around a current-carrying conductor, conduction and displacement current density, derivation of Maxwell's equation from Ampere's law, Faraday's law and Gauss's law. Wave equation derived from Maxwell's equation is presented in chapter five, while chapter six looks into divergence and stoke's theorems. This chapter comprises time harmonic waves, poynting vector (s), media of plane waves, wavelength of wave in free space, wave propagation in free space and in perfect dielectric, etc. Transmission line is presented in chapter seven and lastly, chapter eight contains miscellaneous problems and their solutions.*

This book tries to build a good background for the targeted students and set them to the context of electromagnetism without over assumption. To this end, a student of electromagnetism, studying for the basic principles of and building blocks for communication and power system engineering, will find this text an invaluable material and relief for hitherto dreaded but all important course in Electrical and Electronics Engineering.

Happy Reading!

Akinsanmi . O.
Ijemaru . G.K.
February, 2015.

FOREWORD

This book – ***"A Concise Approach to Electromagnetic Fields and Waves"*** covers the required breadth of the subject matter as prescribed by NUC curriculum for engineering students of Universities and Polytechnics. The area on fields particularly opens the study into field theories needed in engineering for proper understanding of basic electrical engineering ideas.

An in-depth background of Electromagnetics as applied to wave theory and practice has been laid. The book is recommended to students not only in electrical and communication engineering but also in physical science.

Prof. Engr. I.I. ENEH

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CONTENTS

CHAPTER ONE

INTRODUCTION TO ELECTROMAGNETIC FIELDS AND WAVES

1.1 BASIC CONCEPTS ABOUT ELECTROMAGNETICS

Electromagnetic (EM) may be regarded as the study of the interactions between electric and magnetic charges at rest and in motion. It entails the analysis, synthesis of physical interpretation, and application of Electric and Magnetic fields.

Electromagnetic therefore is an aspect of Electrical Engineering which deals with Electric and Magnetic phenomena. The phenomena are defined in terms of Fields and Waves.

The question to ask is: **What is a Field?** A field is space or region where forces act. It is a region in which each point is affected by a force. For instance, objects fall to the ground because they are affected by the force of gravitational filed.

Sources of field: Notably, we have two sources of fields, and they are:

(i) Electric source which is the Electric field E
(ii) Magnetic source which is the Magnetic field H

The conglomeration of E and H gives the Electromagnetic Fields. Hence, we say that EM is a branch of physics or electrical engineering in which electric and magnetic phenomena are studied. The electric field is produced by stationary charges and the magnetic field by moving charges. These two are often referred to as the sources of the field.

Wave: A wave is a means of transporting energy and information. A wave motion transfers energy from one point to another, often with no permanent displacement of the particles of the medium. Electromagnetic waves are composed of oscillating electric and magnetic fields at right angles to each other and both are perpendicular to the direction of propagation of the wave. EM waves do not require a material medium for propagation. Examples of electromagnetic wave are Light rays, Radio waves, TV waves, micro waves, X–rays, Gamma rays, visible light, etc.

Electromagnetics forms the basis of the field of Electrical, Electronics and Computer Engineering. EM principles find applications in various allied disciplines such as Microwaves, Antennas, Electric machines, Satellite communications, Bio-electromagnetics, Plasma, Nuclear research, Fiber optics, radio meteorology and remote sensing to mention but a few. Some of the devices found in EM may range from transformers, electric relays, Radio/TV, Telephone, Electric motors, Transmission lines, Wave guards, Lasers, etc. The design of these devices is anchored on the thorough knowledge of the laws and principles of EM.

1.2 THE DISCOVERY OF ELECTROMAGNETIC FIELD (EMF)

Michael Faraday was the first to use the term “field” in 1849. However, James Clark Maxwell started the study of electromagnetic. He was a Scottish physicist and was referred to as the father of electromagnetic. He celebrated the work which led to the discovery of electromagnetic waves. Maxwell published the first unified theory on electricity and magnetism. The theory comprised all previously known results both experimental and theoretical on electricity and magnetism. He further introduced displacement current and predicted the existence of electromagnetic waves.

Maxwell equations were not fully accepted by many scientists until when confirmed by Heinrich Rudolt Hertz who was a German physics professor. Hertz was successful in generating and detecting radio wave through the application of Maxwell’s work.

1.3 MAXWELL’S EQUATIONS

The four notable equations in both dynamic and static fields produced by Maxwell are as follows:

(a) $\nabla . D = \rho$

(b) $\nabla . B = 0$

(c) $\nabla \times E = -\frac{\partial B}{\partial t}$

(d) $\nabla \times H = Jc + \frac{\partial D}{\partial t}$

Static field

(e) $\nabla . D = \rho$

(f) $\nabla . B = 0$

(g) $\nabla \times E = 0$

(h) $\nabla \times H = \mathrm{J}_c$

Where

D = Electric charge density

B = Magnetic charge density

E = Electric field strength or intensity

H = Magnetic field strength or intensity

ρ = Volume charge density

J_c = Current charge density

∇= Vector differential operator

X = Curl

$\nabla.D$ = Divergent of the Electric charge density

The above equations are known as the Differential or Point form of Maxwell's equations

1.4 ELECTROMAGNETIC SPECTRUM

Spectrum refers to the constituent element of the electromagnetic.

EM Phenomena	Application	Approximate Frequency Range
1. Cosmic rays	Astronomy Physics	10^{14}GHz and above
2. Gamma rays	Cancer therapy	$10^{10} – 10^{13}$GHz
3. X – rays	X – ray examination	$10^{8} – 10^{9}$ GHz
4. Ultraviolet radiation	Sterilization	$10^{6} – 10^{8}$ GHz
5. Visible light	Human vision	$10^{5} – 10^{6}$ GHz
6. Infra-red	Photography	$10^{3} – 10^{4}$ GHz
7. Microwave waves	Radar, microwave, relay, satellite	3 – 300 GHz
8. Radio wave	UHF television VHF television, FM radio SW radio, AM radio	54 – 216 MHz 3 – 216 MHz 525 – 1605 KHz

1.5 FUNDAMENTAL CHARACTERISTICS OF EM WAVES

1. They travel with a constant velocity of $3 \times 10^8 \mathrm{m/s}$ in vacuum
2. They assume all the properties of waves such as reflection, refraction, interference, diffraction, and polarization.
3. They radiate outwardly from the source without the benefit of any.

1.6 GENERAL CHARACTERISTICS OF EM WAVES

1. They are transverse in nature.
2. The intensity of EM waves depends on the electrical field strength of the wave.
3. The directions of electric and magnetic fields are perpendicular to each other and both are perpendicular to the direction of wave propagation.
4. EM waves need no material medium for propagation
5. EM waves are propagated by oscillating electric and magnetic fields oscillating at right angles to each other.
6. Fast propagation of EM waves depends only on the electrical properties and the magnetic medium by which it occurs.

CHAPTER TWO

REVIEW OF VECTOR ANALYSIS

INTRODUCTION: Vector Analysis is a mathematical tool with which electromagnetic (EM) concepts are most conveniently expressed and best understood. A quantity can be either a scalar or a vector. A scalar is a quantity that has only magnitude but no direction. Examples include time, mass, distance, charge, temperature, entropy, electric potential, etc. On the other hand, a vector is a quantity that has both magnitude and direction. Examples include velocity, force, displacement, electric field intensity, del operator, motional charges, etc.

2.1 VECTOR ALGEBRA

Fundamental Laws of Vector Algebra: If $\underline{a}$, $\underline{b}$ and $\underline{c}$ are vectors and m and n are scalars, then the following vector equations can hold as true:

1. $\underline{a} + \underline{b} = \underline{b} + \underline{a}$
2. $\underline{a}m = m\underline{a}$
3. $(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c}) = \underline{(a + c)} + \underline{b}$
4. $\underline{a}(m + n) = \underline{a}m + \underline{a}n$
5. $m(\underline{a} + \underline{b}) = \underline{a}m + \underline{b}m$
6. $\underline{a}.\underline{b} = \underline{b}.\underline{a}$ (Commutative law)
7. $\underline{a}.(\underline{b} + \underline{c}) = \underline{a}.\underline{b} + \underline{a}.\underline{c}$ (Distributive law)

Vector Component in 3-D: We represent the 3-dimensional vector a in a rectangular co-ordinate system as:

$\underline{a} = a_1\mathrm{i} + \mathrm{a}_2\mathrm{j} + \mathrm{a}_3\mathrm{k}$ Or $\underline{a} = \mathrm{a}_x\mathrm{i} + \mathrm{a}_y\mathrm{j} + \mathrm{a}_z\mathrm{k}$, where i, j, and k are the unit vectors in the x, y and z directions respectively

Thus, in 3-D, a vector A will be defined by: $\mathrm{A} = \mathrm{A}_{\mathrm{xi}} + \mathrm{A}_{\mathrm{yj}} + \mathrm{A}_{\mathrm{zk}}$. The magnitude of the vector A is defined by:

$|\mathrm{A}| = \sqrt{\mathrm{A}^2{}_{\mathrm{x}} + \mathrm{A}^2{}_{\mathrm{y}} + \mathrm{A}^2{}_{\mathrm{z}}}$ and the unit vector along A is given by: $\mathrm{a}_{\mathrm{A}} = \frac{\mathrm{A}_{\mathrm{x}}\mathrm{a}_{\mathrm{x}} + \mathrm{A}_{\mathrm{y}}\mathrm{a}_{\mathrm{y}} + \mathrm{A}_{\mathrm{z}}\mathrm{a}_{\mathrm{z}}}{\sqrt{\mathrm{A}^2{}_{\mathrm{x}} + \mathrm{A}^2{}_{\mathrm{y}} + \mathrm{A}^2{}_{\mathrm{z}}}}$

Example 2.1: Given that $\mathrm{A} = 3\mathrm{i} - 2\mathrm{j} + \mathrm{k}$ and $\mathrm{B} = 2\mathrm{i} - 4\mathrm{j} - 3\mathrm{k}$, find:

(i) a_{A} and (ii) $|\mathrm{B}|$

When you are done, check your answer in the box below:

(i) $a_A = \frac{3i-2j+k}{\sqrt{14}}$ (ii) $\|B\| = \sqrt{29}$

2.1.1 ADDITION AND SUBTRACTION OF VECTORS

If $\underline{a} = a_1 i + a_2 j + a_3 k$ and $\underline{b} = b_1 i + b_2 j + b_3 k$, then

(i) $\underline{a} + \underline{b} = (a_1 + b_1)i + (a_2 + b_2)j + (a_3 + b_3)k$

(ii) $\underline{a} - \underline{b} = (a_1 - b_1)i + (a_2 - b_2)j + (a_3 - b_3)k$

Example 2.2: Given that $A = 3i - 2j + k, B = 2i - 4j - 3k$ and $C = -i + 2j + 2k$,

Find: (i) $|A + B + C|$ (ii) $|2A - 3B - 5C|$

When you are done with the above problem, confirm your answer with the one in the box

(i) $\|A + B + C\| = 5.7$ (ii) $\|2A - 3B - 5C\| = 5.5$

Example 2.3: A particle is under the action of the following forces given in terms of their components: $F_1 = 2i + 6j - 4k, F_2 = 4i + 2j + 5k, F_3 = 7i - 4j - 3k,$ and $F_4 = -5i - 2j - 3k$. Calculate:

(i) the resultant force on the particle

(ii) the magnitude of the resultant

(iii) the unit vector parallel to the resultant.

Solution:

(i) Total force $F = 8i - 2j - 5k$

(ii) $|F| = \sqrt{93}$

(iii) $\overline{F} = \frac{F}{|F|} = \frac{8i-2j-5k}{\sqrt{93}}$

2.1.2 VECTOR MULTIPLICATION

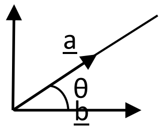
There are two kinds of vector multiplication. They include the scalar (or Dot) product and the vector (or Cross product).

DOT OR SCALAR PRODUCT

The DOT product is obtained when two vectors are multiplied to give a scalar. Hence, the DOT product is a scalar, since the magnitudes of $\underline{a}$ and $\underline{b}$ are scalars. If $\underline{a}$ and $\underline{b}$ are inclined at an angle θ as shown below, then

$$\underline{\mathbf{a}} \cdot \underline{\mathbf{b}} = |\underline{\mathbf{a}}||\underline{\mathbf{b}}| \cos \boldsymbol{\theta}$$

Where θ is the angle between the two vectors and $0 \leq \theta \leq \pi$



The following laws are valid for the DOT product:

(a) $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$ [Commutative law]

(b) $\underline{a} \cdot (\underline{b} + \underline{c}) == \underline{a}.\underline{b} + \underline{a}.\underline{c}$ [Distributive law]

(c) $\underline{a} \cdot \underline{a} = |\underline{a}|^2 = \underline{a}^2$

(d) $\underline{a} \cdot \underline{a} \geq 0$ and $\underline{a} \cdot \underline{a} = 0$ if and only if $\underline{a} = 0$

(e) $i.i = j.j = k.k = 1$

(f) $\mathrm{i.j} = \mathrm{j.k} = \mathrm{k.i} = 0$

(g) If $\underline{a} \cdot \underline{b} = 0$, then $\underline{a}$ and $\underline{b}$ are perpendicular and $\theta = 90^\circ$ so that $\cos\theta = 0$. Hence the two vectors are said to be orthogonal (i.e. neither has a component in the direction of the other.)

(h) If $\underline{a} = (a_x + a_y + a_z)$ and $\underline{b} = (b_x + b_y + b_z)$, then

$\underline{a} \cdot \underline{b} = (a_x b_x) + (a_y b_y) + (a_z b_z)$

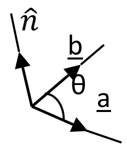
CROSS OR VECTOR PRODUCT

The cross product is obtained when two vectors are multiplied to give a vector. If $\underline{a}$ and $\underline{b}$ are the two vectors, then

$$\underline{\mathbf{a}} \times \underline{\mathbf{b}} = |\underline{\mathbf{a}}||\underline{\mathbf{b}}| \sin\boldsymbol{\theta}\,\hat{\mathbf{n}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} \\ \mathbf{b_1} & \mathbf{b_2} & \mathbf{b_3} \end{vmatrix}$$

$$= (\mathbf{a_2 b_3} - \mathbf{a_3 b_2}) + (\mathbf{a_3 b_1} - \mathbf{a_1 b_3}) + (\mathbf{a_1 b_2} - \mathbf{a_2 b_1})$$

Where $\hat{\mathbf{n}}$ is the unit vector in the plane formed by $\underline{a}$ and $\underline{b}$



The following laws are valid for the Cross product:

(a) $\underline{a} \times \underline{b} \neq \underline{b} \times \underline{a} = -\underline{b} \times \underline{a}$ [anti-commutative]

(b) $\underline{a} \times (\underline{b} + \underline{c}) = (\underline{a} \times \underline{b}) + (\underline{a} \times \underline{c})$ [Distributive]

(c) $i \times i = j \times j = k \times k = 0$ [Parallel]

(d) $i \times j = k,\ j \times k = i,\ k \times i = j$ [Perpendicular]

(e) $j \times i = -k,\ k \times j = -i,\ i \times k = -j$

(f) $\underline{a} \times \underline{a} = 0$

(g) $\underline{a} \times 0 = 0$

(h) $\underline{a} \times \underline{b} = 0$ if and only if $\underline{a}$ and $\underline{b}$ are parallel.

(i) Area of a parallelogram $= |\underline{a} \times \underline{b}|$, where $|\underline{a}|$ and $|\underline{b}|$ are the adjacent sides.

(j) Area of a triangle $\Delta = \frac{1}{2}|\underline{a} \times \underline{b}|$, where $\underline{a} \times \underline{b} = |\underline{a}||\underline{b}| \sin\theta$

TRIPLE PRODUCTS

When two vectors are crossed, the outcome is a vector; but this product can be dotted or crossed further with a third vector to from a triple product.

(a) Scalar Triple Product: The scalar triple product can be defined as:

$\underline{a}\,.(\underline{b} \times \underline{c}) = \underline{b}\,.(\underline{c} \times \underline{a}) = \underline{c}\,.(\underline{a} \times \underline{b}) =$ Scalar result. Hence,

$$\underline{a}.\begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 i\,.(b_2c_3 - b_3c_2)i + a_2 j\,.(b_3c_1 - b_1c_3)j + a_3 k\,.(b_1c_2 - b_2c_1k) = a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$$

$$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

(b) Vector Triple Product: The vector triple product can be defined as:

$\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}(\underline{a}\,.\underline{c}) - \underline{c}(\underline{a}\,.\underline{b}) = vector\ product$

However, $\underline{a} \times (\underline{b} \times \underline{c}) \neq (\underline{a} \times \underline{b}) \times \underline{c}$ [Associative law for cross product fails]

Example 2.4: Given that $\underline{a} = 3i + 2j$ and $\underline{b} = 4i - j$, find the angle between $\underline{a}$ and $\underline{b}$

Solution:

$\underline{a}\,.\underline{b} = |\underline{a}||\underline{b}| \cos\theta$

$\cos\theta = \frac{\underline{a}\,.\underline{b}}{|\underline{a}||\underline{b}|} = \frac{(4i-j)(3i+2j)}{\sqrt{13}.\ \sqrt{17}} = 47.7°$

Example 2.5: Determine the value of p for which the vectors $\underline{a} = 3i + 2j - 4k$ and $\underline{b} = 2i + pj + 5k$ are perpendicular.

Solution:

Since they are perpendicular, $\theta = 90°$ and $\cos 90° = 0$

$\therefore\ \underline{a} . \underline{b} = |\underline{a}||\underline{b}| \cos\theta = 0 \ \rightarrow (3i + 2j - 4k)(2i + pj + 5k) = 0 \rightarrow p = 7$

EXERCISE 2.1

1. Given that; $\underline{a}$ = 2i +3j + 4k, $\underline{b}$ = i + 2j + 3k, $\underline{c}$ = 2i + 4j +k
   Show that: (i) $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} . \underline{c})\underline{b} - (\underline{a} . \underline{b})\underline{c}$
   (ii) $(\underline{a} \times \underline{b}) \times \underline{c} = (\underline{a} . \underline{c})\underline{b} - (\underline{b} . \underline{c})\underline{a}$
2. Given two vectors $\underline{a} = 3i + 12j - 4k$ and $\underline{a} = 6i + 3j + 2k$, find (i) $\underline{a} . \underline{b}$ (ii)θ_{ab}
3. Three field quantities are given by $P = 2a_x - a_z$, $Q = 2a_x - 3a_y + a_z$ and $R = 2a_x - 3a_y + a_z$. Determine: (i) $(P + Q) \times (P - Q)$ (ii) $Q . R \times P$ (iii) $P . Q \times R$ (iv) $P \times (Q \times R)$ (v) a unit vector L_{ar} to both Q and R (vi) the component of P along Q.
4. Show that the vectors $a = (4,0,-1)$, $b = (1,3,4)$ and $c = (-5,-3,-3)$ form the sides of a triangle. Hence, calculate the area of the triangle.
5. The vertices of a triangle are located at $(4,1,-3),,$ and $(0,1,6)$. Find the three angles of the triangle.
6. Show that $(A . B)^2 + (A \times B)^2 = (AB)^2$
7. Find the area of a triangle whose vertices are having the position vectors: $2i + 3j + 4k$, $3i + 4j + 2k$ and $4i + 2j + 3k$
8. If $i - j - 3k$ and $2i - j - 3k$ are the diagonals of a parallelogram, find the area of the parallelogram.
9. Find the projection of the vector $4i - 3j + k$ on the line passing through the point $(2,3 - 1)$ and $(-2,-4,3)$
10. Calculate the magnitude of the moment of the force $F = 8i - 4j + 3k$ about the point $O(2i + 0j + k)$ if the force is applied at the point $P(6i + 2j + 3k)$

2.2 VECTOR CALCULUS

The integration and differentiation of vectors is known as vector calculus. Certain fundamental ideas in electromagnetics and mathematics generally are better expressed with the concept of calculus.

2.2.1 DEL OPERATOR (∇)

The Del Operator, ∇, is the vector differential operator.

$$\nabla = \frac{\partial}{\partial x}a_x + \frac{\partial}{\partial y}a_y + \frac{\partial}{\partial z}a_z = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

The vector differential operator, otherwise known as the gradient operator, is not a vector in itself, but when it operates on a scalar function, a vector ensues. The operator can be used to define the following:

a. The gradient of a scalar, V, written as ∇V
b. The divergence of a vector, A, written as $\nabla . A$
c. The curl of a vector, A, written as $\nabla \times A$
d. The Laplacian of a scalar, V, written as $\nabla^2 V$

2.2.2 GRADIENT OF A SCALAR

The gradient of a scalar field ∅ is a vector, denoted by

$$\text{grad } \emptyset = \nabla\emptyset = \frac{\partial\emptyset}{\partial x}a_x + \frac{\partial\emptyset}{\partial y}a_y + \frac{\partial\emptyset}{\partial z}a_z = i\frac{\partial\emptyset}{\partial x} + j\frac{\partial\emptyset}{\partial y} + z\frac{\partial\emptyset}{\partial z}$$

Example 2.6: If $\emptyset = r^2 = x^2 + y^2 + z^2$ and $\underline{r} = xi + yi + zk$, find $\nabla\emptyset$

Solution:

$$\nabla\emptyset = i\frac{\partial\emptyset}{\partial x} + j\frac{\partial\emptyset}{\partial y} + z\frac{\partial\emptyset}{\partial z} = i\frac{\partial}{\partial x}(x^2 + y^2 + z^2) + j\frac{\partial}{\partial y}(x^2 + y^2 + z^2) + k\frac{\partial}{\partial z}(x^2 + y^2 + z^2)$$

$$\nabla\emptyset = 2xi + 2yj + 2zk = 2(xi + yj + zk) = 2r \rightarrow \nabla\emptyset = \nabla r^2 = 2r, \text{ since } \emptyset = r^2.$$

2.2.3 DIVERGENCE OF A VECTOR

The divergence of a function $F = f_1i + f_2j + f_3k$ is expressed as:

$$\nabla . F = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right).(F_1i + F_2j + F_3k)$$

$= \frac{\partial}{\partial x}\mathrm{F}_1 + \frac{\partial}{\partial x}\mathrm{F}_2 + \frac{\partial}{\partial x}\mathrm{F}_3$

The following properties of divergence should be noted.

a. The divergence of a vector field produces a scalar field
b. The divergence of a scalar field has no meaning
c. $\nabla.(\mathrm{A}+\mathrm{B}) = \nabla.\mathrm{A} + \nabla.\mathrm{B}$
d. $\nabla.(\mathrm{VA}) = \mathrm{V}\nabla - \mathrm{A} + \mathrm{A}.\nabla\mathrm{V}$

Example 2.7: Determine the divergence for which $\mathrm{A} = \mathrm{x}^2\mathrm{yi} - 2(\mathrm{xy}^2 + \mathrm{y}^3\mathrm{z}) + 3\mathrm{y}^2\mathrm{z}^2\mathrm{k}$

The student is expected to try the above problem. When you are done, check your answer with the one in the box below.

$\nabla.\mathrm{A} = 0$

2.2.4 CURL OF A VECTOR

The curl operator denoted by $\nabla\times$, acts on a vector and gives another vector as a result.

If $\mathrm{A} = \mathrm{a}_x\mathrm{i} + \mathrm{a}_y\mathrm{j} + \mathrm{a}_z\mathrm{k}$, then $\nabla\times\mathrm{A} = \left(\mathrm{i}\frac{\partial}{\partial \mathrm{x}} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times \left(\mathrm{a}_x\mathrm{i} + \mathrm{a}_y\mathrm{j} + \mathrm{a}_z\mathrm{k}\right)$

$$\nabla\times\mathrm{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = \mathrm{i}\left(\frac{\partial}{\partial \mathrm{y}}\mathrm{a}_z - \frac{\partial}{\partial \mathrm{z}}\mathrm{a}_y\right) + \mathrm{j}\left(\frac{\partial}{\partial \mathrm{z}}\mathrm{a}_x - \frac{\partial}{\partial \mathrm{x}}\mathrm{a}_z\right) + \mathrm{k}\left(\frac{\partial}{\partial \mathrm{x}}\mathrm{a}_y - \frac{\partial}{\partial \mathrm{y}}\mathrm{a}_x\right)$$

Curl is thus a vector function. If $\nabla\times\mathrm{A} = 0$, then the vector A is said to be irrotational. The curl of a vector field provides the maximum value of the circulation of the field per unit area, and indicates the direction along which this maximum value occurs. Some of the fundamental properties of a curl of a vector field include:

a. The curl of a vector field is another vector field
b. The curl of a scalar field makes no sense.
c. The divergence of the curl of a vector field vanishes. That is $\nabla.(\nabla\times\mathrm{A}) = 0$
d. The curl of the gradient of a scalar field vanishes. That is $\nabla\times\nabla\emptyset = 0$

e. $\nabla \times (A \times B) = A(\nabla . B) - B(\nabla . A) + (B . \nabla A - (A . \nabla)B)$

f. $\nabla \times (\emptyset A) = \emptyset \nabla \times A + \nabla \emptyset \times A$

Example 2.8: Given vector A = 3i + xj + yk, Find the curl of the vector.

Solution:

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3 & x & y \end{vmatrix}$$

$$\nabla \times A = \left[\frac{\partial}{\partial y}(y) - \frac{\partial}{\partial z}(x)\right] i + \left[\frac{\partial}{\partial z}(3) - \frac{\partial}{\partial x}(y)\right] j + \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(3)\right] k = i + 0j + k = i + k$$

Example 2.9: Given that $A = (y^4 - x^2z^2)i + (x^2 + y^2)j - x^2yzk$, find the curl of the vector A at the point $(1, 0, 2)$.

The student is expected to play around the above problem. At the end, check whether your solution matches with that in the box below.

$\nabla \times A = 2i + 4j + 2k$

2.2.5 DIRECTIONAL DERIVATIVES

Directional derivative = $\nabla \emptyset . T$ or $\nabla \emptyset . \underline{u}$ where $T = \frac{a}{|\underline{a}|} = \underline{u} = \hat{a} =$ Unit Vector

$\nabla \emptyset =$ gradient of $\emptyset$

Example 2.10: Find the directional derivative of the function $\emptyset = x^2z + 2xy^2 + yz^2$ at the point $(1, 2, -1)$ in the direction of the vector A = 2i + 3j – 4k.

Solution:

$$\nabla \emptyset = \frac{\partial}{\partial x}\emptyset i + \frac{\partial}{\partial y}\emptyset j + \frac{\partial}{\partial z}\emptyset k$$

$$= \frac{\partial}{\partial x}(x^2z + 2xy^2 + yz^2)i + \frac{\partial}{\partial y}(x^2z + 2xy^2 + yz^2)j + \frac{\partial}{\partial z}(x^2z + 2xy^2 + yz^2)k$$

$$\nabla \emptyset = (2xz + 2y^2)i + (4xy + z^2)j + (x^2 + 2yz)k$$

At point (1,2, – 1)

$\nabla\emptyset = (-2+8)i + (8+1)j + (1-4)k$

= 6i + 9j – 3k

The unit vector $\hat{a} = \frac{a}{|\underline{a}|} = \frac{2i+3j-4k}{\sqrt{2^2+3^2+(-4)^2}} = \frac{2i+3j-4k}{\sqrt{29}}$

Therefore Directional derivative $\nabla\emptyset.\hat{a} = \frac{1}{\sqrt{29}}(2i+3j-4k).(6i+9j-3k)$

$= \frac{1}{\sqrt{29}}(12+27+12) = \frac{51}{\sqrt{29}} = 9.47$

2.2.6 LAPLACIAN OPERATOR ∇^2

The Laplacian operator ∇^2 is obtained by operating on a scalar field $\emptyset(r)$ with the gradient and divergence operators in turn to form another scalar field div grad $\emptyset$. It should be noted that the Laplacian of a scalar field is another scalar field.

The Laplacian operator is expressed as;

$$\nabla^2 = \nabla.\nabla = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right).\left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

For a scalar u,

$$\nabla^2 u = \frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u + \frac{\partial^2}{\partial z^2}u - Laplacian\ operation\ of\ u$$

For a vector $\underline{u}$,

$$\nabla^2 \underline{u} = \frac{\partial^2}{\partial x^2}\underline{u} + \frac{\partial^2}{\partial y^2}\underline{u} + \frac{\partial^2}{\partial z^2}\underline{u}$$

Example 2.11: If V = $2x^2y - xz^3$. Find the Laplacian operator of V.

Solution:

$$\nabla^2 V = \frac{\partial^2}{\partial x^2}V + \frac{\partial^2}{\partial y^2}V + \frac{\partial^2}{\partial z^2}V = \frac{\partial^2}{\partial x^2}(2x^2y - xz^3) + \frac{\partial^2}{\partial y^2}(2x^2y - xz^3) + \frac{\partial^2}{\partial z^2}(2x^2y - xz^3)$$

= 4y – 6xz

EXERCISE 2.2

1. Given that $V = x^2y^2 + xyz$, compute ∇V and the directional derivative in the direction $3a_x + 4a_y + 12a_z$ at $(2, -1, 0)$.
2. Given that $\emptyset = xy + yz + xz$, find the gradient $\emptyset$ at the point $(1, 2, 3)$ and the directional derivative of $\emptyset$ at the same point in the direction toward point $3, 4, 4)$.
3. Determine the divergence of the vector field P given that $P = x^2yza_x + xza_z$
4. If $A = x^2yi + yz^3j + zx^3k$, determine $\text{grad}(\text{div}A)$
5. If $\emptyset = xyz - 2y^2z + x^2z^2$, determine the div grad $\emptyset$ at the point $(2, 4, 1)$
6. If $A = x^2yzi + xyz^2j + y^2zk$, determine the Curl Curl A at the point $(2, 1, 1)$.

2.2.7 DIV, GRAD, CURL IN CYLINDRICAL AND SPERICAL POLAR COORDINATES

The cylindrical coordinates is represented as (ρ,ф,z) where,

ρ = radius of cylinder

ф = Azimuthal angle measured from the x – axis in the xy plane

z = the same as in Cartesian system.

$0 < \rho < \infty$

$0° < \phi < 2\pi$

$-\infty \leq z \leq +\infty$

Then, the unit vector are $\overrightarrow{a\rho}$, $\overrightarrow{a\theta}$ and $\overrightarrow{aZ}$ respectively. Hence

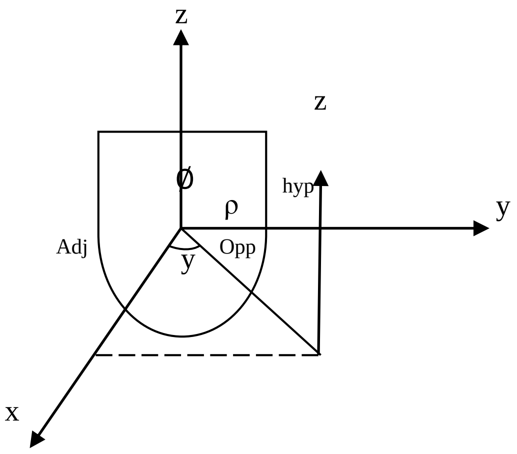
$$\overrightarrow{a\rho}.\overrightarrow{a\rho} = \overrightarrow{a\theta}.\overrightarrow{a\theta} = \overrightarrow{aZ}.\overrightarrow{aZ} = 1$$

$$\overrightarrow{a\rho}.\overrightarrow{a\theta} = \overrightarrow{a\theta}.\overrightarrow{aZ} = \overrightarrow{aZ}.\overrightarrow{a\rho} = 0$$

$$\overrightarrow{a\rho} \times \overrightarrow{a\theta} = \overrightarrow{aZ} \qquad \overrightarrow{a\rho} = \hat{\rho}$$

$$\overrightarrow{a\theta} \times \overrightarrow{aZ} = \overrightarrow{a\rho} \qquad \overrightarrow{a\theta} = \hat{\theta} \qquad \overrightarrow{a\emptyset} = \hat{\emptyset}$$

$$\overrightarrow{aZ} \times \overrightarrow{a\rho} = \overrightarrow{a\theta} \qquad \overrightarrow{aZ} = z$$



Transforming a point from Cartesian to cylindrical (x,y,z) = (ρ,Ø,z)

By Pythagoras theorem

$\rho = \sqrt{x^2 + y^2}$

$\emptyset = \tan^{-1}\left({}^{y}/_{x}\right), z = z$

Transforming from cylindrical to Cartesian coordinates

$\cos\emptyset = {}^{x}/_{\rho}$; $\sin\emptyset = {}^{y}/_{\rho}$; $z = z$

Element of the surface ds = $\rho d\emptyset dz\hat{n}$

Where $\hat{n}$=(cos Ø , sin Ø , 0 Surface differential (differentiate twice)

Element of volume;

dv = ρdρdØdz Volume differential (differentiate thrice)

GRAD

$$\nabla = \frac{\partial}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial\emptyset}\widehat{\emptyset} + \frac{\partial}{\partial z}\hat{z}$$

$$\nabla F = \frac{\partial}{\partial\rho}F\hat{\rho} + \frac{\partial}{\partial\emptyset}F\widehat{\emptyset} + \frac{\partial}{\partial z}F\hat{z}$$

DIVERGENCE

$$\nabla.\mathrm{F} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho F_\rho\right) + \frac{1}{\rho}\frac{\partial}{\partial\emptyset}F_\emptyset + \frac{\partial}{\partial z}F_Z$$

CURL

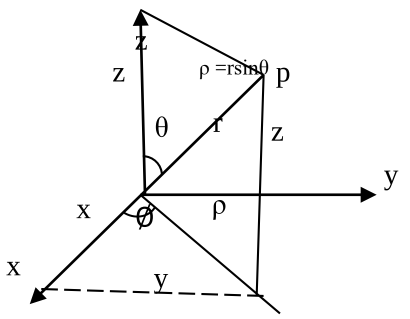
$$\nabla \times F = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\emptyset} & z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\emptyset} & \frac{\partial}{\partial z} \\ F\rho & \rho F_\emptyset & Fz \end{vmatrix}$$

$$= \frac{1}{\rho}\left(\frac{\partial F_Z}{\partial\emptyset} - \rho\frac{\partial F_\emptyset}{\partial z}\right)\hat{\rho} + \frac{1}{\rho}\left(\frac{\partial F\rho}{\partial z} - \frac{\partial Fz}{\partial\rho}\right)\hat{\emptyset} + \frac{1}{\rho}\left(\frac{\partial F_\emptyset}{\partial\rho} - \frac{\partial F\rho}{\partial\emptyset}\right)\hat{z}$$

LAPLACIAN

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\emptyset} + \frac{\partial^2 V}{\partial z^2}$$

SPHERICAL POLAR CO-ORDINATES



P(x,y,z) = P(ρ,∅,z)= P(r,θ, ∅)

By Pythagoras theorem

$$r = \sqrt{\rho^2 + z^2}$$

$$= \sqrt{x^2 + y^2 + z^2}$$

Since $\rho = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1} {}^{p}/_{z} = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z}$

$\emptyset = \tan^{-1} {}^{y}/_{x}$

$\cos\emptyset = {}^{x}/_{\rho}$; $\sin\emptyset = {}^{y}/_{\rho}$; $\sin\theta = {}^{p}/_{r}$

$\Rightarrow x = \rho\cos\emptyset$; $y = p\sin\emptyset$; $\rho = r\sin\theta$

But $\rho = r\sin\theta$

$x = r\sin\theta\cos\emptyset$

$y = r\sin\theta\sin\emptyset$

$\cos\theta = {}^{z}/_{r} \Rightarrow z = r\cos\theta$

$\therefore x = r\sin\theta\cos\emptyset$; $y = r\sin\theta\sin\emptyset$; $z = r\cos\theta$

Where r for the spherical coordinates is the radius from the origin to the point p

θ = angle between the z-axis and the positive vector of P.

∅= azimuthal angle measured from the x-axis in the xy plane.

$0 \le r \le \infty$; $0 \le \theta \le \pi$; $0 \le \emptyset \le 2\pi$

$\overrightarrow{a_r}.\overrightarrow{a_r} = \overrightarrow{a_\theta}.\overrightarrow{a_\theta} = \overrightarrow{a_\emptyset}.\overrightarrow{a_\emptyset} = 1$

$\overrightarrow{a_r}.\overrightarrow{a_\theta} = \overrightarrow{a_\theta}.\overrightarrow{a_\emptyset} = \overrightarrow{a_\emptyset}.\overrightarrow{a_r} = 0$

$\overrightarrow{a_r} \times \overrightarrow{a_\theta} = \overrightarrow{a_\emptyset}$

$\overrightarrow{a_\theta} \times \overrightarrow{a_\emptyset} = \overrightarrow{a_r}$

$\overrightarrow{a_\emptyset} \times \overrightarrow{a_r} = \overrightarrow{a_\theta}$

ds = rsinθdθd∅$\hat{n}$ displacement in surface

dv = r^2sinθdrdθd∅ displacement in volume

GRAD

$$\nabla = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{rsin\theta}\frac{\partial}{\partial \emptyset}\hat{\emptyset}$$

$$\nabla.\mathrm{V} = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{rsin\theta}\frac{\partial V}{\partial \emptyset}\hat{\emptyset}$$

Where $\hat{r} = \overrightarrow{a_r}$; $\hat{\theta} = \overrightarrow{a_\theta}$; $\hat{\emptyset} = \overrightarrow{a_\emptyset}$

DIVERGENCE

$$\nabla\mathrm{V} = \frac{1}{\mathrm{r}^2}\frac{\partial}{\partial r}(r^2V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(V_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \emptyset}\emptyset$$

CURL

$$\nabla\times\mathrm{V} = \frac{1}{\mathrm{r}^2\sin\theta}\begin{vmatrix} \overrightarrow{a_r} & \mathrm{r}\overrightarrow{a_\theta} & \mathrm{r}\sin\theta\,\overrightarrow{a_\theta} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \emptyset} \\ \mathrm{V_r} & \mathrm{rV_\theta} & \mathrm{r}\sin\theta\,\mathrm{V}_\emptyset \end{vmatrix}$$

LAPLACE EQUATION

$$\nabla^2V = \frac{1}{\mathrm{r}^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\mathrm{r}^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{\mathrm{r}^2\sin^2\theta}\left(\frac{\partial^2 V}{\partial \emptyset^2}\right)$$

Summarily,

DIVERGENCE

For a vector A

$\nabla.A = \frac{\partial}{\partial x}Ax + \frac{\partial}{\partial y}Ay + \frac{\partial}{\partial z}Az$ Cartesian

$\nabla.A = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \emptyset}(\rho A_\emptyset) + \frac{\partial}{\partial z}A_z$ Cylinderical

$\nabla.A = \frac{1}{\mathrm{r}^2}\frac{\partial}{\partial r}(r^2Ar) + \frac{1}{\mathrm{r}^2\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{\mathrm{r}\sin\theta}\frac{\partial}{\partial \emptyset}A_\emptyset$ Spherical

CURL

$$\nabla \times A = \begin{vmatrix} \overrightarrow{a_x} & \overrightarrow{a_y} & \overrightarrow{a_z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Ax & Ay & Az \end{vmatrix}$$ Cartesian

$$\nabla \times A = \frac{1}{\rho} \begin{vmatrix} \overrightarrow{a_\rho} & \rho\overrightarrow{a_\emptyset} & \overrightarrow{a_z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \emptyset} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\emptyset & A_z \end{vmatrix}$$ Cylindrical

$$\nabla \times A = \frac{1}{\mathrm{r^2 \sin \theta}} \begin{vmatrix} \overrightarrow{a_r} & r\overrightarrow{a_\emptyset} & r\sin\theta\,\overrightarrow{a_\emptyset} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \emptyset} \\ A_r & rA_\theta & r\sin\theta\, A_\emptyset \end{vmatrix}$$ Spherical

LAPLACIAN OPERATOR

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$ Cartesian

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \emptyset^2} + \frac{\partial^2 V}{\partial z^2}$$ Cylinderical

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\mathrm{r^2 \sin \theta}}\frac{\partial}{\partial \theta}\left[\sin\theta\frac{\partial V}{\partial \theta}\right] + \frac{1}{\mathrm{r^2 \sin^2 \theta}}\frac{\partial^2 V}{\partial \emptyset^2}$$ Spherical

Example 2.12: Find the gradient of the following scalars

(a) $V = e^{-z} \sin 2x \cos hy$

(b) $U = \rho^2 z \cos 2\emptyset$

(c) $W = 10r\sin^2\theta\cos\theta$

Solution:

(a) $\nabla V = \frac{\partial}{\partial x}Vi + \frac{\partial}{\partial y}Vj + \frac{\partial}{\partial z}Vk$

$=2e^{-z} \cos 2\mathrm{x} \cosh y\, i + e^{-z} \sin 2x \sinh y\, j - e^{-z} \sin 2x \text{ coshyk}$

(b) $U = \rho^2 z \cos 2\emptyset$

$$\nabla U = \frac{\partial}{\partial \rho}u\overrightarrow{a_\rho} + \frac{1}{\rho}\frac{\partial}{\partial \emptyset}u\overrightarrow{a_\emptyset} + \frac{\partial}{\partial z}u\overrightarrow{a_z}$$

$$= \frac{\partial}{\partial \rho}(\rho^2 z \cos 2\emptyset)\overrightarrow{a_\rho} + \frac{1}{\rho}\frac{\partial}{\partial \emptyset}(\rho^2 z \cos 2\emptyset)\overrightarrow{a_\emptyset} + \frac{\partial}{\partial z}(\rho^2 z \cos 2\emptyset)\overrightarrow{a_z}$$

$= 2pz\cos 2\emptyset\,\overrightarrow{a_\rho} - 2pz\sin 2\emptyset\overrightarrow{a_\emptyset} + \rho^2 z\cos 2\emptyset\,\overrightarrow{a_z}$

(c) W = 10rsin^{2}θcosθ

$$\nabla W = \frac{\partial}{\partial r}W\overrightarrow{a_r} + \frac{1}{r}\frac{\partial}{\partial\theta}W\overrightarrow{a_\emptyset} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\emptyset}W\overrightarrow{a_\emptyset}$$

$$= \frac{\partial}{\partial r}(10\sin^2\theta\cos\emptyset)\overrightarrow{a_r} + \frac{1}{r}\frac{\partial}{\partial\theta}(10\sin^2\theta\cos\emptyset)\overrightarrow{a_\emptyset} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\emptyset}(10\sin^2\theta\cos\emptyset)\overrightarrow{a_\emptyset}$$

$$10\sin^2\theta\cos\emptyset\,\overrightarrow{a_r} + 10\sin\theta\cos\theta\cos\emptyset\,\overrightarrow{a_\emptyset} - 10\sin\theta\sin\emptyset\,\overrightarrow{a_\emptyset}$$

EXERCISE 2.3

(1) Determine the divergence of the following vector

(a) $x^2yz\overrightarrow{a_x} + xz\overrightarrow{a_z}$

(b) $\rho\sin\emptyset\,\overrightarrow{a_\rho} + \rho^2 z\overrightarrow{a_\emptyset} + z\cos\emptyset\,\overrightarrow{a_z} = R$

(2) Determine the Laplacian of the scalar field

(a) $V = e^{-z}\sin 2x\cosh y$

(b) $V = \rho^2 z\cos 2\emptyset$